# A Technique to Classify the Similarity Solutions of Nonlinear Partial (Integro-)Differential Equations. II. Full Optimal Subalgebraic Systems

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"Optimal systems" of similarity solutions of a given system of nonlinear partial (integro-)differential equations which admits a finite-dimensional Lie point symmetry group $G$ are an effective systematic means to classify these group-invariant solutions since every other such solution can be derived from the members of the optimal systems. The classification problem for the similarity solutions leads to that of "constructing" optimal subalgebraic systems for the Lie algebra $\mathscr{G}$ of the known symmetry group $G$. The methods for determining optimal systems of $s$-dimensional Lie subalgebras up to the dimension $r$ of $\mathscr{G}$ vary in case of $3 \leq s \leq r$, depending on the solvability of $\mathscr{G}$. If the $r$-dimensional Lie algebra $\mathscr{G}$ of the infinitesimal symmetries is nonsolvable, in addition to the optimal subsystems of solvable subalgebras of $\mathscr{G}$ one has to determine the optimal subsystems of semisimple subalgebras of $\mathscr{G}$ in order to construct the full optimal systems of $s$-dimensional subalgebras of $\mathscr{G}$ with $3 \leq s \leq r$. The techniques presented for this classification process are applied to the nonsolvable Lie algebra $\mathscr{G}$ of the eight-dimensional Lie point symmetry group $G$ admitted by the three-dimensional Vlasov-Maxwell equations for a multi-species plasma in the non-relativistic case.

*Key words:* Group theory; Statistical plasma physics.

## Introduction

Similarity analysis is a powerful tool for obtaining exact similarity solutions of (nonlinear) partial differential equations (PDEs). In this paper we assume that the reader is familiar with most of the required theory of these applications of Lie groups to PDEs, which are systematically and well described in the textbooks of Ovsiannikov [1], Ibragimov [2], Bluman and Cole [3], Olver [4], Bluman and Kumei [5], and Stephani [6]. The inclusion of integrodifferential equations (IPDEs) in the Lie group method was carried out by some authors, especially Taranov [7], Marsden [8], Tajiri [9], and Roberts [10], and we refer the reader to the mentioned papers for detailed information in this case.

As mentioned in my previous paper [11] (further referred as I), the main aims of the similarity analysis of a given system $\mathscr{F}$ of PDEs (or IPDEs) in $n$ independent and $m$ dependent real variables are to calculate and to classify the similarity solutions of $\mathscr{F}$. First one has to determine the maximal Lie point symmetry group $G$ admitted by $\mathscr{F}$. This group $G$ consists of all the real valued transformations acting on an open and connected subset $M$ of the Euclidean space $\mathbb{R}^{n+m}$ which leave the system $\mathscr{F}$ invariant and map (graphs of) solutions to (graphs of)solutions. The required theory and description of the techniques to determine this connected local transformation group $G$ can be found in the mentioned books (see especially Ovsiannikov [1] and Olver [4]).

In what follows, we assume that $G$ is a known $r$-parameter symmetry group admitted by $\mathscr{F}$, where $r$ is a natural number. A similarity solution of the $s$-parameter subgroup $H$ of $G$ ($H$-invariant solution of $\mathscr{F}$) is a solution of $\mathscr{F}$ whose graph is invariant relative to the elements of $H$ and may be obtained, under additional regularity assumptions on the action of $H$ on $M$, by solving a reduced system of PDEs (IPDEs) with $n-s$ independent and $m$ dependent variables (see e.g. Olver [4]). It is not usually feasible to list all possible similarity solutions of all the $s$-parameter subgroups, since the number of these subgroups is almost always infinite, and to each $s$-parameter subgroup there will correspond a family of group-invariant solutions. Therefore, one desires to minimize the search for similarity solutions by listing the essentially different group-invariant solutions, which leads to the concept of an optimal system of similarity solutions from which every other such solution can be derived. The

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concept is based on the following result (see Olver [4]): For any $g \in G$ with $g \notin H$ a $H$-invariant solution is transformed to a $g \cdot H \cdot g^{-1}$-invariant solution, where the two subgroups $H$ and $g \cdot H \cdot g^{-1}$ are called conjugate. If $H$ and $H'$ denote two $s$-parameter subgroups of $G$, then a $H$-invariant solution and a similarity solution of the subgroup $H'$ are called essentially different if $H$ and $H'$ are not conjugate, i.e. there exists no element $g \in G$ such that $H' = g \cdot H \cdot g^{-1}$. Thus, the classification problem for the similarity solutions of $s$-parameter subgroups of $G$ leads to that of separating the collection of all $s$-parameter subgroups of $G$ into conjugacy classes of subgroups. A minimal list of non-conjugate $s$-parameter subgroups (one from every equivalence class) with $1 \le s \le r$ is said to be an optimal system $\Theta_s^G$ for the symmetry group $G$. In order to classify the similarity solutions of $\mathscr{F}$ one is interested in optimal systems $\Theta_s^G$ with $1 \le s \le \min(r, n)$.

The problem of finding an optimal system $\Theta_s^G$ for $G$ is equal to that of finding an optimal system $\Theta_s^{\mathscr{G}}$ of $s$-dimensional subalgebras for the $r$-dimensional real Lie algebra $\mathscr{G}$ of the symmetry group $G$, where the Lie algebra of $G$ is identified with the isomorphic Lie algebra $\mathscr{G}$ of the Killing vector fields on $M$ whose flows coincide with the actions of the one-parameter subgroups of $G$ on $M$ (see e.g. Olver [4]). Here, a list of $s$-dimensional subalgebra forms an optimal system $\Theta_s^{\mathscr{G}}$ if every $s$-dimensional subalgebra $\mathscr{H}$ of $\mathscr{G}$ is conjugate to a unique member $\mathscr{K}$ of the list under some element of the adjoint representation, i.e. there is an inner automorphism $\mathbf{Ad}(g): \mathscr{G} \to \mathscr{G}$ with $g \in G$ such that $\mathbf{Ad}(g)(\mathscr{H}) = \mathscr{K}$.

A detailed description of most of the known techniques for the construction of optimal subalgebraic systems for the Lie algebra of infinitesimal symmetries admitted by a system of PDEs can be found in the textbooks of Ovsiannikov [1], Ibragimov [2], and Olver [4], where also a few examples are given. Following Ovsiannikov [1] and Ibragimov [2], Galas [12] formulated an "algorithm" for the determination of optimal systems of $s$-dimensional subalgebras of an arbitrary real Lie algebra $\mathscr{G}$ of infinitesimal symmetries for $1 \le s \le r$, where the classification process varies, depending on the solvability of $\mathscr{G}$ in case of $3 \le s \le r$. For a nonsolvable Lie algebra $\mathscr{G}$ the problem of classifying its $s$-dimensional subalgebras with $3 \le s \le r$ is generally harder (see below).

In a series of papers (cf. [13], [14], and [15]) Patera, Sharp, Winternitz, and Zassenhaus developed another, but related method of classifying the subalgebras of a real (complex) finite-dimensional Lie algebra under conjugation and applied it to some fundamental Lie algebras of physics. Especially for the Poincaré algebra (see [13] and also [16]) these authors gave a complete classification of its subalgebras up to dimension five relative to the group of complex valued inner automorphisms of the algebra.

Recently, Coggeshall and Meyer-ter-Vehn [17] determined the 14-parameter Lie point symmetry group admitted by the three-dimensional, one-temperature hydrodynamic equations, including conduction and a thermal source. (Related systems were investigated by Ovsiannikov [1], Coggeshall and Axford [18], Coggeshall [19]). For a seven-parameter solvable subgroup corresponding to two-dimensional axisymmetric geometry they calculated optimal systems of one- and two-dimensional subalgebras of the Lie algebra of this subgroup using the techniques described by Ovsiannikov [1] and Galas [12].

In I a survey of these known techniques for the construction of optimal subsystems of solvable subalgebras for the real Lie algebra $\mathscr{G}$ of an arbitrary finite-dimensional Lie point symmetry group is given. Furthermore, in I a modified method, which is based on the properties of the bilinear invariant forms (relative to the local Lie group of the inner automorphisms of the Lie algebra $\mathscr{G}$ or relative to the groups of the inner automorphisms of its subalgebras), is presented, where the calculation of these invariant bilinear (symmetric) forms may be done using computer-algebra programs, exspecially the REDUCE 3.2 programs OPTSYS and BINV, which are also described in I. The knowledge of the associated "linear" and "bilinear" invariants of an arbitrary vector in $\mathscr{G}$ is helpful during the process of classifying the solvable and nonsolvable subalgebras of $\mathscr{G}$, since these invariants cannot be changed by the full adjoint action. The advantage of this modified technique is shown in I by applying it to the nine-dimensional solvable Lie algebra of infinitesimal symmetries admitted by the two-dimensional non-stationary ideal magnetohydrodynamic equations.

In case of a solvable $r$-dimensional real Lie algebra $\mathscr{G}$, the optimal subsystems $\tilde{\Theta}_s^{\mathscr{G}}$ of solvable $s$-dimensional subalgebras of $\mathscr{G}$ with $1 \le s \le r$ are full optimal systems $\Theta_s^{\mathscr{G}}$, since any subalgebra of $\mathscr{G}$ is also solvable (see e.g. lemma 1 in I). Since the techniques for obtaining optimal subalgebraic systems for a solvable Lie algebra or optimal subsystems of solvable subalgebras of an arbitrary real finite-dimensional Lie algebra have been discussed in I, we restrict our attention in the present work to the problem of classifying the $s$-dimensional subalgebras of an arbitrary nonsolvable $r$-dimensional real Lie algebra $\mathscr{G}$ by means of full optimal subalgebraic systems $\Theta_s^{\mathscr{G}}$ with $3 \leq s \leq r$. A description of the techniques for this classification process based on the investigations of Ibragimov [2] is given in Section 1. In addition to the optimal subsystems $\tilde{\Theta}_s^{\mathscr{G}}$ of $s$-dimensional solvable subalgebras, one has to determine optimal subsystems $\hat{\Theta}_s^{\mathscr{L}}$ of the $s$-dimensional semisimple subalgebras of a Levi subalgebra $\mathscr{L}$ of $\mathscr{G}$ and optimal subsystems $\hat{\Theta}_s^{\mathscr{G}}$ of $s$-dimensional subalgebras with non-trivial Levi decompositions.

The problem of determining optimal subsystems $\Theta_s^{\mathscr{L}}$ for a semisimple Levi factor $\mathscr{L}$ of $\mathscr{G}$ leads to the classification theory of complex semisimple Lie algebras and their real forms, which is the great achievement of the classical work of Cartan and Killing and is treated in most books on Lie algebras (see e.g. Varadarajan [20]. Since in course of similarity analysis of PDEs (IPDEs) one frequently finds three-dimensional real Levi subalgebras, we only investigate these cases.

In Sect. 2 the three-dimensional (3-D) Vlasov-Maxwell equations (VMS) for a multi-species plasma in the non-relativistic case are regarded. This system admits an eight-parameter real Lie point symmetry group $G$ with a nonsolvable Lie algebra $\mathscr{G}$. The group $G$ contains the three-dimensional Euclidean group E(3), which is the semidirect product of the rotation group SO(3, $\mathbb{R}$) and the group of space translations T(3), the one-parameter group of time translation, and one scaling group. Using the techniques described in I and in Sect. 1, we determine full optimal systems of $s$-dimensional subalgebras of the eight-dimensional Lie algebra $\mathscr{G}$ for $s = 1, \dots, 8$. Here, the (bi-)linear invariants of an arbitrary vector in the Lie algebra $\mathscr{G}$ (relative to the group of inner automorphisms of the Lie algebra $\mathscr{G}$) are rather helpful to obtain optimal subalgebraic systems, since with the aid of these invariants it is possible to shorten the lengthy classification process in comparison with the usual techniques.

## 1. Optimal Systems for Nonsolvable Lie Algebras

In this section we describe the basic tools for the construction of full optimal systems of $s$-dimensional subalgebras for the real Lie algebra $\mathscr{G}$ of a given $r$-parameter Lie point symmetry group $G$ admitted by a system of partial (integro-)differential equations, where $s \in \mathbb{N}$ and $r \in \mathbb{N}$ denote natural numbers. Here, we restrict most of our attention to the task of classifying the $s$-dimensional subalgebras for a nonsolvable Lie algebra $\mathscr{G}$ in case $3 \leq s \leq r$, since the techniques for obtaining optimal subsystems of solvable subalgebras of a nonsolvable Lie algebra $\mathscr{G}$ are summarized in I. The following description of the available techniques is based on the textbooks of Ovsiannikov [1] and Ibragimov [2], which Galas [12] used to formulate an "algorithm" for the determination of optimal subalgebraic systems without using the invariants of the group of the inner automorphisms. These invariants can play an important role during the construction of optimal subalgebraic systems (see e.g. Sect. 1 in I). For the proofs of the theorems stated below we refer the reader to the usual books on Lie groups and Lie algebras, for example Jacobson [21], Sagle and Walde [22], Varadarajan [20], or Hilgert and Neeb [23].

In what follows, we denote the dimension of the real finite-dimensional Lie algebra $\mathscr{G}$ by $\mathrm{d}(\mathscr{G}) := \dim(\mathscr{G}) = r \in \mathbb{N}$ and assume that a basis of $\mathscr{G}$ is given by $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_r\}$. Then any $g$ in a neighbourhood of the identity element of the Lie point symmetry group $G$ with Lie algebra $\mathscr{G}$ may be written as

$$g = \exp(\varepsilon_1 \boldsymbol{v}_1) \cdot \ldots \cdot \exp(\varepsilon_r \boldsymbol{v}_r), \tag{1}$$

where $\cdot$ denotes the group multiplication in $G$ and the coefficients $\varepsilon_1, \dots, \varepsilon_r \in \mathbb{R}$ are called canonical coordinates of ther second kind (see Hilgert and Neeb [23]). A full optical system $\Theta_s^{\mathscr{G}}$ of $s$-dimensional subalgebras of $\mathscr{G}$ with $s \leq r$ is defined as the union of the representatives of conjugate (or similar) algebra classes of given dimension $s$ (one from every class), where two $s$-dimensional subalgebras $\mathscr{H}$ and $\mathscr{K}$ are called conjugate if there exists an inner automorphism $\mathbf{Ad}(g) \in \mathrm{Int}(\mathscr{G})$ $(g \in G)$ such that $\mathbf{Ad}(g)(\mathscr{H}) = \mathscr{K}$. Here, the general inner automorphism in the local Lie group $\mathrm{Int}(\mathscr{G})$ consisting of all inner automorphisms $\mathscr{G} \to \mathscr{G}$ may be given by

$$\begin{aligned}\mathbf{Ad}(g = \exp(\varepsilon_1 \boldsymbol{v}_1) \cdot \ldots \cdot \exp(\varepsilon_r \boldsymbol{v}_r)) \\ = \mathbf{Ad}(\exp(\varepsilon_1 \boldsymbol{v}_1) \circ \ldots \circ \mathbf{Ad}(\exp(\varepsilon_r \boldsymbol{v}_r)),\end{aligned} \tag{2}$$

where the linear mapping $\mathbf{Ad}(g = \exp(\varepsilon \boldsymbol{v})): \mathscr{G} \to \mathscr{G}$ maps the vector $\boldsymbol{w} \in \mathscr{G}$ to the Lie series

$$\begin{aligned}\mathbf{Ad}(\exp(\varepsilon \boldsymbol{v}))(\boldsymbol{w}) = \exp(\varepsilon\, \mathbf{ad}(\boldsymbol{v}))(\boldsymbol{w}) = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} (\mathbf{ad}(\boldsymbol{v}))^k(\boldsymbol{w}) \\ = \boldsymbol{w} + \varepsilon [\boldsymbol{w}, \boldsymbol{v}] + \frac{\varepsilon^2}{2} [[\boldsymbol{w}, \boldsymbol{v}], \boldsymbol{v}] + \ldots,\end{aligned} \tag{3}$$

and the inner derivation $\mathbf{ad}(v)$ for a given $v \in \mathcal{G}$ is defined by

$$\mathbf{ad}(v)\colon \mathcal{G} \to \mathcal{G}; \quad w \mapsto \mathbf{ad}(v)(w) := [w, v] \tag{4}$$

(see Ovsiannikov [1] and Olver [4]). Clearly, $\Theta_r^{\mathcal{G}}$ consists only of the Lie algebra $\mathcal{G}$ itself.

Since all one- and two-dimensional Lie algebras are solvable, the optimal subalgebraic systems $\Theta_1^{\mathcal{G}}$ and $\Theta_2^{\mathcal{G}}$ for an arbitrary real finite-dimensional Lie algebra $\mathcal{G}$ consist only of solvable Lie subalgebras and can be constructed using the techniques described e.g. in Sect. 1 in I. Therefore $\Theta_j^{\mathcal{G}} = \tilde{\Theta}_j^{\mathcal{G}}$ holds for $j = 1, 2$. For a solvable real $r$-dimensional Lie algebra $\mathcal{G}$, any of its subalgebras is solvable (cf. lemma 1 in I) and an optimal subalgebraic system $\Theta_{s+1}^{\mathcal{G}}$ with $1 \le s \le (r-1)$ may be obtained by the "method of expansion" (see subsection 1.2 in I) of the representative members of a known optimal system $\Theta_s^{\mathcal{G}}$, since, according to the Lie theorem in I, every $(s+1)$-dimensional solvable algebra contains an $s$-dimensional solvable subalgebra. This still holds when one looks for the solvable subalgebras of an arbitrary real Lie algebra $\mathcal{G}$ in order to obtain optimal subsystems $\tilde{\Theta}_s^{\mathcal{G}}$ of solvable subalgebras for $\mathcal{G}$ (see e.g. Ibragimov [2] and I).

To classify subalgebras of dimensionality $2 < s \le r$ in the case of a nonsolvable real $r$-dimensional Lie algebra $\mathcal{G}$, one can sucessfully employ the Levi-Mal'cev theorem asserting the existence and uniqueness (up to conjugation) of the Levi decomposition of $\mathcal{G}$ into a semidirect sum of its radical and a semisimple Levi factor (see e.g. Ibragimov [2]). Furthermore, the Mal'cev-Harish-Chandra theorem gives an important addition to the Levi-Mal'cev theorem such that most subalgebras of a nonsolvable $\mathcal{G}$ can be completely classified by means of both theorems (see e.g. Ovsiannikov [1] and Ibragimov [2]). In this paper, we restrict our attention to the task of classifying the $s$-dimensional subalgebras of a nonsolvable real $r$-dimensional Lie algebra with a non-trivial Levi decomposition in case $3 \le s \le r$. First we give some basic definitions and theorems, which are necessary to state the Levi-Mal'cev theorem and the Mal'cev–Harish-Chandra theorem and to proceed to the description of the techniques available for the named classification problem.

**Theorem 1** (see e.g. Ovsiannikov [1] and Varadarajan [20]). *Among the solvable (nilpotent) ideals of a finite-dimensional Lie algebra $\mathcal{G}$ there is a unique solvable ideal $\mathcal{R}(\mathcal{G})$ (nilpotent ideal $\mathcal{N}(\mathcal{G})$) of maximal dimension, which contains any solvable (nilpotent) ideal of $\mathcal{G}$. This maximal solvable ideal $\mathcal{R}(\mathcal{G})$ (nilpotent ideal $\mathcal{N}(\mathcal{G})$) in $\mathcal{G}$ is called the radical (nilradical) of $\mathcal{G}$. Then $\mathcal{N}(\mathcal{G}) \subset \mathcal{R}(\mathcal{G})$ and $[\mathcal{R}(\mathcal{G}), \mathcal{G}] \subset \mathcal{N}(\mathcal{G})$.*

Hence, the radical of a finite-dimensional solvable Lie algebra $\mathcal{G}$ is $\mathcal{G}$ itself, i.e. $\mathcal{R}(\mathcal{G}) = \mathcal{G}$, and its derived algebra $\mathcal{G}^{(1)} := [\mathcal{G}, \mathcal{G}]$ is a subalgebra of its nilradical $\mathcal{N}(\mathcal{G})$. The radical of any Lie algebra is invariant under all (inner) automorphisms of the Lie algebra (see e.g. Varadarajan [20]).

**Definition 1** (see e.g. Ibragimov [2]). *A noncommutative finite-dimensional Lie algebra $\mathcal{G}$, i.e. $\mathcal{G}^{(1)} \neq 0$, is said to be simple if it has no ideals different from the null algebra 0 and $\mathcal{G}$ (0 consisting of the null vector and $\mathcal{G}$ being called the trivial ideals in $\mathcal{G}$). A finite-dimensional Lie algebra $\mathcal{G}$ is called semisimple if its radical reduces to zero, i.e. $\mathcal{R}(\mathcal{G}) = 0$.*

From theorem 1 and the above definition it follows that the factor algebra $\mathcal{G}/\mathcal{R}(\mathcal{G})$ is semisimple for a finite-dimensional Lie algebra $\mathcal{G}$. Since one- and two-dimensional Lie algebras are solvable, i.e. the radical of such a Lie algebra is the algebra itself, there exists no semisimple Lie algebra $\mathcal{G}$ of dimension $\mathrm{d}(\mathcal{G}) = 1$ or $\mathrm{d}(\mathcal{G}) = 2$. In the course of similarity analysis of PDEs (IPDEs) one frequently encounters three-dimensional semisimple subalgebras of the real Lie algebra $\mathcal{G}$ of infinitesimal symmetries (see e.g. Ibragimov [2] and Ovsiannikov [1]). These semisimple real Lie algebras of dimensionality 3 must be isomorphic either to the Lie algebra $\mathcal{SO}(3, \mathbb{R})$ of the rotation group $SO(3, \mathbb{R})$ or to the Lie algebra $\mathcal{SL}(2, \mathbb{R})$ of all real $2 \times 2$-matrices with trace zero, where both algebras are real forms of the complex Lie algebra $\mathcal{SL}(2, \mathbb{C})$ (see e.g. Jacobson [21]). Any three-dimensional real Lie algebra not containing two-dimensional subalgebras is isomorphic to the Lie algebra of rotations $\mathcal{SO}(3, \mathbb{R})$ with the basis $\{u_1, u_2, u_3\}$ and the structure $[u_1, u_2] = u_3$, $[u_2, u_3] = u_1$, $[u_3, u_1] = u_2$ (see e.g. Ovsiannikov [1]). The special linear algebra $\mathcal{SL}(2, \mathbb{R})$ spanned by the three basis vectors $w_1$, $w_2$, $w_3$ has the structure $[w_1, w_3] = 2w_1$, $[w_2, w_3] = -2w_2$, $[w_1, w_2] = w_3$ and contains two-dimensional subalgebras (see Jacobson [21]). Both $\mathcal{SO}(3, \mathbb{R})$ and $\mathcal{SL}(2, \mathbb{R})$ are even simple Lie algebras. Instead of the above definition one can use the following criterion of semisimplicity of a finite-dimensional real Lie algebra $\mathcal{G}$.

**Cartan's Criterion of Semisimplicity** (see e.g. Ovsiannikov [1]). *The $r$-dimensional real Lie algebra $\mathcal{G}$ with basis $\{v_1, \ldots, v_r\}$ is semisimple if and only if the*

*Killing form* $K_{\mathscr{G}}: \mathscr{G} \times \mathscr{G} \to \mathbb{R}$, *which maps the pair* $(\boldsymbol{u}, \boldsymbol{v})$ *of the two vectors* $\boldsymbol{u} = \sum_{j=1}^{r} \alpha_j \boldsymbol{v}_j \in \mathscr{G}$, $\boldsymbol{v} = \sum_{k=1}^{r} \beta_k \boldsymbol{v}_k \in \mathscr{G}$ *with* $\alpha_j, \beta_k \in \mathbb{R}$ $(j, k = 1, \ldots, r)$ *to*

$$K_{\mathscr{G}}(\boldsymbol{u}, \boldsymbol{v}) := \operatorname{tr}(\mathbf{ad}(\boldsymbol{u}) \circ \mathbf{ad}(\boldsymbol{v})) =: \sum_{j,k=1}^{r} \alpha_j K_{jk}^{\mathscr{G}} \beta_k ,$$

*is nondegenerate, i.e.* $\det(K_{jk}^{\mathscr{G}}) \neq 0$.

We call the real symmetric matrix $K^{\mathscr{G}} := (K_{jk}^{\mathscr{G}})$ the Killing matrix (relative to the given basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ of the $r$-dimensional Lie algebra $\mathscr{G}$).

The radical of a finite-dimensional Lie algebra can be characterized with the aid of the Killing form by the following theorem.

**Theorem 2** (see e.g. Ovsiannikov [1]). *The radical* $\mathscr{R}(\mathscr{G})$ *of the finite-dimensional Lie algebra* $\mathscr{G}$ *is a set of elements* $\boldsymbol{u} \in \mathscr{G}$ *for which* $K_{\mathscr{G}}(\boldsymbol{u}, \boldsymbol{v}) = 0$ *for any element* $\boldsymbol{v}$ *of the derived algebra* $\mathscr{G}^{(1)} := [\mathscr{G}, \mathscr{G}]$.

Using theorem 2, the REDUCE 3.2 program OPTSYS (see I) determines the radical of the finite-dimensional real Lie algebra $\mathscr{G}$ of infinitesimal symmetries.

**Definition 2** (see e.g. Ibragimov [2]). *Let* $\mathscr{G}$ *be a Lie algebra. If* $\mathscr{I}$ *is an ideal in* $\mathscr{G}$ *and* $\mathscr{J}$ *is a subalgebra (ideal) of* $\mathscr{G}$ *with* $\mathscr{I} \cap \mathscr{J} = 0$ *(and* $[\mathscr{I}, \mathscr{J}] = 0$*), then the direct sum of the subspaces* $\mathscr{I}$ *and* $\mathscr{J}$ *of the vector space* $\mathscr{G}$ *forms a Lie algebra, which is called the semidirect sum (direct sum) of the subalgebras* $\mathscr{I}$ *and* $\mathscr{J}$ *of* $\mathscr{G}$, *denoted by* $\mathscr{I} \oplus_s \mathscr{J}$ $(\mathscr{I} \oplus \mathscr{J})$. *If, in addition,* $\mathscr{I} \oplus_s \mathscr{J} = \mathscr{G}$ $(\mathscr{I} \oplus \mathscr{J} = \mathscr{G})$, *then we say that* $\mathscr{G}$ *is the semidirect sum (direct sum) of* $\mathscr{I}$ *and* $\mathscr{J}$.

The classification of a semisimple Lie algebra is based on the following theorem.

**Structural Theorem** (see e.g. Ovsiannikov [1] or Hilgert and Neeb [23]). *A finite-dimensional real Lie algebra* $\mathscr{G}$ *is semisimple if and only if there exist simple ideals* $\mathscr{G}_1, \ldots, \mathscr{G}_k$ $(k \in \mathbb{N})$ *of* $\mathscr{G}$ *such that*

$$\mathscr{G} = \mathscr{G}_1 \oplus \mathscr{G}_2 \oplus \ldots \oplus \mathscr{G}_k .$$

*The only ideals of* $\mathscr{G}$ *are the direct sums* $\mathscr{I} = \sum_{j \in J} \mathscr{G}_j$ *with* $J \subset \{1, \ldots, k\}$.

With the aid of this theorem it is established that every ideal of a semisimple Lie algebra is semisimple, and for a semisimple Lie algebra $\mathscr{G}^{(1)} = \mathscr{G}$. In essence, this theorem reduces the study of semisimple Lie algebras to that of simple algebras, which is well treated in most books on Lie algebras (see e.g. Varadarajan [20]).

The Levi theorem shows that any finite-dimensional Lie algebra decomposes in the semidirect sum of its radical and a semisimple subalgebra.

**Levi Theorem** (see e.g. Ibragimov [2] and Varadarajan [20]). *Let* $\mathscr{G}$ *be a finite-dimensional Lie algebra* $\mathscr{G}$ *and* $\mathscr{R}(\mathscr{G})$ *its unique radical. Then* $\mathscr{G}$ *is the semidirect sum*

$$\mathscr{G} = \mathscr{R}(\mathscr{G}) \oplus_s \mathscr{L} \tag{5}$$

*of* $\mathscr{R}(\mathscr{G})$ *and a semisimple subalgebra* $\mathscr{L}$ *of* $\mathscr{G}$, *which is not uniquely determined.* (5) *is called a Levi decomposition and* $\mathscr{L}$ *is said to be a Levi subalgebra or Levi factor. If* $\mathscr{L}$ *is a Levi subalgebra of* $\mathscr{G}$, *then it is also a Levi factor of the derived algebra* $\mathscr{G}^{(1)}$ *of* $\mathscr{G}$ *and* $\mathscr{G}^{(1)} = [\mathscr{R}(\mathscr{G}), \mathscr{G}] \oplus_s \mathscr{L}$ *is a Levi decomposition of* $\mathscr{G}^{(1)}$.

The Levi decomposition is trivial for semisimple Lie algebras ($\mathscr{R}(\mathscr{G}) = 0$) and solvable Lie algebras ($\mathscr{R}(\mathscr{G}) = \mathscr{G}$). The uniqueness (up to conjugation) of a Levi decomposition for a finite-dimensional Lie algebra is proved by the Mal'cev theorem, which is a corollary of the following Mal'cev–Harish-Chandra theorem.

**Mal'cev–Harish-Chandra Theorem** (see e.g. Ovsiannikov [1] and Varadarajan [20]). *Let* $\mathscr{G}$ *be a finite-dimensional real Lie algebra,* $\mathscr{R}(\mathscr{G})$ *its radical, and* $\mathscr{N}(\mathscr{G})$ *its nilradical. If* $\mathscr{G} = \mathscr{R}(\mathscr{G}) \oplus_s \mathscr{L}$ *is a Levi decomposition of* $\mathscr{G}$ *with a semisimple Levi factor* $\mathscr{L}$, *then for any semisimple subalgebra* $\mathscr{M}$ *of* $\mathscr{G}$ *there exists* $\boldsymbol{w} \in \mathscr{G}$ $\langle \boldsymbol{w} \in [\mathscr{R}(\mathscr{G}), \mathscr{G}] \subset \mathscr{N}(\mathscr{G}) \rangle$ *such that the inner automorphism* $\exp(\mathbf{ad}(\boldsymbol{w})) \in \operatorname{Int}(\mathscr{G})$ $\cdot \langle \exp(\mathbf{ad}(\boldsymbol{w})) \in \operatorname{Int}([\mathscr{R}(\mathscr{G}), \mathscr{G}]) \rangle$ *takes* $\mathscr{M}$ *into* $\mathscr{L}$.

For the sake of simplicity we stated the above theorem in a weak form (see e.g. Ovsiannikov [1]). In addition, we gave the strong form in parentheses $\langle \ldots \rangle$ (see Varadarajan [20]).

**Corollary 1** (see e.g. Jacobson [21]). *Any semisimple subalgebra of a finite-dimensional real Lie algebra* $\mathscr{G}$ *can be imbedded in a Levi factor of* $\mathscr{G}$.

Hence, every semisimple subalgebra of a finite-dimensional real Lie algebra $\mathscr{G}$ is conjugate to a subalgebra of any Levi factor of $\mathscr{G}$. If $\mathscr{H} = \mathscr{R}' \oplus_s \mathscr{L}'$ is an arbitrary Levi decomposition for a subalgebra $\mathscr{H}$ of a finite-dimensional real Lie algebra $\mathscr{G}$, where $\mathscr{R}' = \mathscr{R}(\mathscr{H})$ is the radical of $\mathscr{H}$ and $\mathscr{L}'$ is a Levi subalgebra of $\mathscr{H}$, then $\mathscr{R}'$ and $\mathscr{L}'$ are a solvable and a semisimple subalgebra of $\mathscr{G}$, respectively. Thus, the

Levi factor $\mathscr{L}'$ of $\mathscr{H}$ has to be conjugate to a semisimple subalgebra in any Levi subalgebra of $\mathscr{G}$.

**Mal'cev Theorem** (see e.g. Jacobson [21] or Ibragimov [2]). *Let $\mathscr{G}$ be a finite-dimensional real Lie algebra and $\mathscr{R}(\mathscr{G})$ its radical. If there are two Levi decompositions $\mathscr{G} = \mathscr{R}(\mathscr{G}) \oplus_s \mathscr{L} = \mathscr{R}(\mathscr{G}) \oplus_s \mathscr{L}'$ of $\mathscr{G}$, then there is a $w \in \mathscr{R}(\mathscr{G})$ such that the inner automorphism $\exp(\mathbf{ad}(w))$ takes $\mathscr{L}'$ into $\mathscr{L}$.*

The following corollary of the Mal'cev theorem shows that the classification problem for semisimple subalgebras of a finite-dimensional real Lie algebra $\mathscr{G}$ is equivalent to the task of classifying the semisimple subalgebras of a Levi factor $\mathscr{L}$ of $\mathscr{G}$, considered independently of $\mathscr{G}$.

**Corollary 2** (see e.g. Ibragimov [2]). *Let $\mathscr{G}$ be a finite-dimensional real Lie algebra, $\mathscr{R}(\mathscr{G})$ its radical, and $\mathscr{L}$ a Levi factor. Let $\mathscr{M}$ and $\mathscr{N}$ be two conjugate semisimple subalgebras of the Levi subalgebra $\mathscr{L}$, i.e. there is an inner automorphism $\mathbf{Ad}(g) \in \mathrm{Int}(\mathscr{G})$ such that $\mathscr{M} = \mathbf{Ad}(g)(\mathscr{N})$. Then there exists an inner automorphism $\exp(\mathbf{ad}(w)) \in \mathrm{Int}(\mathscr{L})$ with $w \in \mathscr{L}$ such that $\mathscr{M} = \exp(\mathbf{ad}(w))(\mathscr{N})$.*

### *1.1 "Construction" of Full Optimal Subalgebraic Systems*

In what follows, we describe an "algorithm" to construct optimal systems of subalgebras for an arbitrary $r$-dimensional real Lie algebra $\mathscr{G}$ with null center and the fixed basis $\{v_1, \ldots, v_r\}$.

*Remark.* If the Lie algebra $\mathscr{G}$ contains a nonzero center $\mathscr{Z}(\mathscr{G}) := \{w \in \mathscr{G} \mid [u, w] = 0 \; \forall u \in \mathscr{G}\} \not\equiv 0$, then it is sufficient to classify the subalgebras of the factor algebra $\mathscr{G}/\mathscr{Z}(\mathscr{G})$ by means of optimal subalgebraic systems, where the factor algebra $\mathscr{G}/\mathscr{Z}(\mathscr{G})$ is isomorphic to the Lie algebra $\mathbf{ad}(\mathscr{G})$ consisting of all the inner derivations (see e.g. Hilgert and Neeb [23]). If full optimal systems of subalgebras are known for $\mathscr{G}/\mathscr{Z}(\mathscr{G})$ or for $\mathbf{ad}(\mathscr{G})$, then they can be considered as known for the entire Lie algebra $\mathscr{G}$ (see e.g. Ovsiannikov [1] and I).

During the construction of $\Theta_1^{\mathscr{G}}$ consisting of conjugacy classes of one-dimensional subalgebras of $\mathscr{G}$ one begins with the selection of a nonnull vector $u = \sum_{j=1}^{r} \alpha_j v_j$ with $\alpha_j \in \mathbb{R}$ $(j = 1, \ldots, r)$ and its image $w := \mathbf{Ad}(g)(v) = \sum_{k=1}^{r} \beta_k v_k$ under the general inner automorphism $\mathbf{Ad}(g) \in \mathrm{Int}(\mathscr{G})$ given in (2), where $g = \exp(\varepsilon_1 v_1) \cdot \ldots \cdot \exp(\varepsilon_r v_r)$ with real parameters $\varepsilon_1, \ldots, \varepsilon_r$. The values of these parameters should be choosen to achieve the maximum possible "simplification" of the real coordinates $\beta_1, \ldots, \beta_r$ of $w$. This permits the choice of the simplest representative of the class of conjugate subalgebras to which the Lie algebra spanned by $u$ belongs, where the ("linear" and "bilinear") invariants of $u$ (relative to $\mathrm{Int}(\mathscr{G})$) place restrictions on how far one can expect to "simplify" the vector $w \in \mathscr{G}$ (see e.g. Olver [4] or I). The various possibilities of selecting $u \in \mathscr{G}$ give the conjugacy classes of one-dimensional subalgebras and, from them, an optimal system $\Theta_1^{\mathscr{G}}$ (see e.g. Ovsiannikov [1] and I).

In order to construct an optimal system $\Theta_2^{\mathscr{G}}$ it is possible to use the "method of expansion" of the representative members of a known optimal subalgebraic system $\Theta_1^{\mathscr{G}}$ as described in I (see also Ovsiannikov [1]). Since any one- or-two-dimensional Lie algebra is solvable, full optimal systems $\Theta_1^{\mathscr{G}}$ and $\Theta_2^{\mathscr{G}}$ for $\mathscr{G}$ are given by optimal subsystems $\tilde{\Theta}_1^{\mathscr{G}}$ and $\tilde{\Theta}_2^{\mathscr{G}}$ of solvable subalgebras of $\mathscr{G}$, where $\mathscr{G}$ may be solvable or not.

If $\mathscr{G}$ is solvable, full optimal subalgebraic systems $\Theta_s^{\mathscr{G}}$ are optimal subsystems $\tilde{\Theta}_s^{\mathscr{G}}$ of solvable subalgebras for $\mathscr{G}$ for any $s = 1, \ldots, r$, since any subalgebra of $\mathscr{G}$ is solvable. Any optimal subalgebraic systems $\Theta_{j+1}^{\mathscr{G}}$ for $1 \leq j \leq (r-1)$ may be constructed using the method of expansion of the members of an optimal system $\Theta_j^{\mathscr{G}}$ (see e.g. subsection 1.2 in I).

From now on, let the $r$-dimensional real Lie algebra $\mathscr{G}$ be nonsolvable. If the radial $\mathscr{R}(\mathscr{G})$ of $\mathscr{G}$ and a Levi decomposition $\mathscr{G} = \mathscr{R}(\mathscr{G}) \oplus_s \mathscr{L}$ are known, then the construction of fully optimal systems $\Theta_s^{\mathscr{G}}$ of $s$-dimensional subalgebras with $3 \leq s \leq r$ may be done successively in the following way (cf. Ibragimov [2] and Galas [12]):

**Step 1 (Optimal subsystems $\tilde{\Theta}_s^{\mathscr{G}}$ of solvable subalgebras).** According to the construction of optimal subsystems of solvable subalgebras (see subsection 1.2 in I), one determines an optimal subsystem $\tilde{\Theta}_s^{\mathscr{G}}$ of all the nonconjugate $s$-dimensional solvable subalgebras of the Lie algebra $\mathscr{G}$ using the method of expansion for the representatives of an optimal subsystem $\tilde{\Theta}_{s-1}^{\mathscr{G}}$.

**Step 2 (Optimal subsystems $\hat{\Theta}_s^{\mathscr{L}}$ of semisimple subalgebras).** Following corollary 2, one determines in case of $s < \mathrm{d}(\mathscr{L})$ all the nonconjugate $s$-dimensional semisimple subalgebras of the semisimple Levi subalgebra $\mathscr{L}$. If $s = \mathrm{d}(\mathscr{L})$, the only semisimple subalgebra is the Levi factor $\mathscr{L}$ itself. If $s$ is less than, or equal to, the dimension $\mathrm{d}(\mathscr{L})$ of $\mathscr{L}$, an optimal subsystem $\hat{\Theta}_s^{\mathscr{L}}$ of $s$-dimensional semisimple subalgebras of the Levi factor $\mathscr{L}$ has to be constructed, which is empty if there is no $s$-dimensional semisimple subalgebra in $\mathscr{L}$. Any semisimple subalgebra in $\mathscr{L}$. Any $s$-dimensional semisimple subalgebra in $\mathscr{G}$ is then conjugate to one of the representatives in $\hat{\Theta}_s^{\mathscr{L}}$.

**Step 3 (Optimal subsystems $\hat{\Theta}_s^{\mathscr{G}}$ of subalgebras with non-trivial Levi decompositions).** For any existing conjugacy class of the $j$-dimensional semisimple subalgebras of the Levi factor $\mathscr{L}$ in an optimal system $\hat{\Theta}_j^{\mathscr{L}}$ with $3 \le j < s$ and $j \le \mathrm{d}(\mathscr{L})$ one chooses a representative semisimple subalgebra $\mathscr{L}'$ of $\mathscr{G}$ and determines all the $(s-j)$-dimensional solvable subalgebras $\mathscr{R}'$ of $\mathscr{G}$ such that $\mathscr{R}' \cap \mathscr{L}' = 0$ and $[\mathscr{R}', \mathscr{L}'] \subset \mathscr{R}'$ hold. The resulting $s$-dimensional subalgebras of type $\mathscr{H} = \mathscr{R}' \oplus_s \mathscr{L}'$ are listed. With the aid of this list one constructs an optimal subsystem $\hat{\Theta}_s^{\mathscr{G}}$ of these $s$-dimensional subalgebras. Any other $s$-dimensional subalgebra of $\mathscr{G}$ with non-trivial Levi decomposition is conjugate to a representative of a class in $\hat{\Theta}_s^{\mathscr{G}}$.

**Step 4 (Full optimal subalgebraic systems $\Theta_s^{\mathscr{G}}$).** The union of the three optimal subsystems $\tilde{\Theta}_s^{\mathscr{G}}$, $\hat{\Theta}_s^{\mathscr{L}}$ and $\hat{\Theta}_s^{\mathscr{G}}$ obtained in this way forms a full optimal subalgebraic system $\Theta_s^{\mathscr{G}}$ for $\mathscr{G}$.

In case of $s=3$ a full optimal subalgebraic system $\Theta_3^{\mathscr{G}}$ for $\mathscr{G}$ is given by $\tilde{\Theta}_3^{\mathscr{G}}$ and $\hat{\Theta}_3^{\mathscr{L}}$. If the dimension of the Levi factor $\mathscr{L}$ is $\mathrm{d}(\mathscr{L}) = 3$, then $\mathscr{G}$ decomposes into the semidirect sum of its radical and $\mathscr{L}$, where the Levi subalgebra is isomorphic either to $\mathscr{SO}(3, \mathbb{R})$ or to $\mathscr{SL}(2, \mathbb{R})$. The Lie algebra of infinitesimal symmetries admitted by the three-dimensional non-relativistic Vlasov-Maxwell system contains $\mathscr{SO}(3, \mathbb{R})$ as a three-dimensional Levi factor (see Section 2).

Ovsiannikov [1] constructed in his example § 14.9 full optimal subalgebraic systems for a four-dimensional reductive Lie algebra $\mathscr{G}$ with a three-dimensional Levi factor $\mathscr{L}$ isomorphic to the special linear algebra $\mathscr{SL}(2, \mathbb{R})$, where a Lie algebra is said to be reductive if its radical coincides with its center (see Varadarajan [20]). Since $\mathscr{G}$ is reductive, the Lie algebra $\mathscr{G}$ is the direct sum $\mathscr{Z}(\mathscr{G}) \oplus \mathscr{G}^{(1)}$ (see e.g. Varadarajan [20]) of the one-dimensional center $\mathscr{Z}(\mathscr{G})$ and the semisimple derived algebra $\mathscr{G}^{(1)} = \mathscr{L}$ isomorphic to $\mathscr{SL}(2, \mathbb{R})$.

In the course of similarity analysis of a system of PDEs (or IPDEs) one sometimes finds a finite-dimensional real Lie algebra $\mathscr{I}$ of the infinitesimal symmetries of the given system, which is an ideal $\mathscr{I}$ in an arbitrary real Lie algebra $\mathscr{G}$ for which optimal subalgebraic systems are known. Thus, the question arises whether the task of classifying the subalgebras for the Lie algebra $\mathscr{I}$ under consideration may be simplified with the aid of the known optimal subalgebraic systems for $\mathscr{G}$. In what follows, we describe a technique for this classification problem, where we assume that $\mathscr{I}$ is a proper $\tilde{r}$-dimensional ideal of the real $r$-dimensional Lie algebra $\mathscr{G}$ such that $0 < \mathrm{d}(\mathscr{I}) = \tilde{r} < \mathrm{d}(\mathscr{G}) = r$ and the factor algebra $\mathscr{G}/\mathscr{I}$ has a small dimension $0 < \mathrm{d}(\mathscr{G}/\mathscr{I}) = r - \tilde{r}$. The latter condition of small dimensionality of the factor algebra is stated for the sake of practicability of the techniques given below. The basic results for the construction of optimal systems for $\mathscr{I}$ are: If $\mathscr{I}$ is a proper ideal of $\mathscr{G}$, then $\exp(\mathbf{ad}(\boldsymbol{v}))(\mathscr{I}) = \mathscr{I}$ holds for any inner automorphism $\exp(\mathbf{ad}(\boldsymbol{v})) \in \mathrm{Int}(\mathscr{G})$ with $\boldsymbol{v} \in \mathscr{G}$. If the two $s$-dimensional subalgebras $\mathscr{H}$ and $\mathscr{K}$ of the ideal $\mathscr{I}$ in $\mathscr{G}$ are nonconjugate (relative to $\mathrm{Int}(\mathscr{G})$), they cannot be conjugate relative to the group $\mathrm{Int}(\mathscr{I})$. If full optimal subalgebraic systems $\Theta_s^{\mathscr{G}}$ (relative to $\mathrm{Int}(\mathscr{G})$) are known for $1 \le s \le \tilde{r}$, then a complete set of optimal systems $\Theta_s^{\mathscr{I}}$ (relative to $\mathrm{Int}(\mathscr{I})$) for $s = 1, \ldots, \tilde{r}$ can be constructed in the following way (cf. Galas [12]):

We define the list $\vartheta_s^{\mathscr{I}}$ as the union of the non-conjugate $s$-dimensional subalgebras in $\Theta_s^{\mathscr{G}}$ (one representative from every conjugacy class relative to $\mathrm{Int}(\mathscr{G})$), which are also subalgebras of the Lie algebra $\mathscr{I}$. Furthermore, for an arbitrary subalgebra $\mathscr{H} \in \vartheta_s^{\mathscr{I}}$, let $\tilde{\mathfrak{J}}_s^{\mathscr{I}}(\mathscr{H})$ be the set of all subalgebras $\exp(\mathbf{ad}(\tilde{\boldsymbol{v}}))(\mathscr{H})$, where $\tilde{\boldsymbol{v}}$ ranges over a set of representatives of all cosets $\tilde{\boldsymbol{v}} + \mathscr{I} \in \mathscr{G}/\mathscr{I}$. The union of the sets $\tilde{\mathfrak{J}}_s^{\mathscr{I}}(\mathscr{H})$ for all $\mathscr{H} \in \vartheta_s^{\mathscr{I}}$ is denoted by $\tilde{\mathfrak{J}}_s^{\mathscr{I}}$. Successively for any $s$-dimensional subalgebra $\mathscr{H}$ of $\mathscr{I}$ in $\vartheta_s^{\mathscr{I}}$ one determines the conjugacy classes in $\tilde{\mathfrak{J}}_s^{\mathscr{I}}(\mathscr{H})$ relative to $\mathrm{Int}(\mathscr{I})$ and lists one representative from every class. Then the union of these lists forms a full optimal system $\Theta_s^{\mathscr{I}}$ (relative to $\mathrm{Int}(\mathscr{I})$) for $\mathscr{I}$. In general, it follows that $\vartheta_s^{\mathscr{I}} \subset \Theta_s^{\mathscr{I}} \subset \tilde{\mathfrak{J}}_s^{\mathscr{I}}$. If $\mathscr{G}$ is the direct sum of the ideal $\mathscr{I}$ and an ideal $\mathscr{K}$, i.e. $\mathscr{G} = \mathscr{I} \oplus \mathscr{K}$, then even $\vartheta_s^{\mathscr{I}} = \Theta_s^{\mathscr{I}} = \tilde{\mathfrak{J}}_s^{\mathscr{I}}$, since $\exp(\mathbf{ad}(\tilde{\boldsymbol{v}}))(\mathscr{H}) = \mathscr{H}$ holds for any $\tilde{\boldsymbol{v}} \in \mathscr{K}$ and for any subalgebra $\mathscr{H}$ of $\mathscr{I}$.

Examples for this construction of optimal systems with the aid of known optimal subalgebraic systems are demonstrated by Galas [12] and Fuchs [24].

## 2. The Three-Dimensional Vlasov-Maxwell Equations

We consider three-dimensional motions of a collisionless plasma of $\sigma$ species without a background or external fields in the non-relativistic case ($\sigma \in \mathbb{N}$). In what follows, the particles of species $\alpha$ have charge $q_\alpha^*$, mass $m_\alpha^*$, and the three-dimensional distribution function $f_1^{\alpha*} = f_1^{\alpha*}(\vec{x}^*, \vec{w}^*, t^*)$, where $\vec{x}^* = x^*\vec{e}_1 + y^*\vec{e}_2 + z^*\vec{e}_3$ is the space vector, $\vec{w}^*$ denotes the velocity vector, and $t^*$ is the time. The distribution functions $f_1^{\alpha*}$ ($\alpha = 1, \ldots, \sigma$) for the $\sigma$ species, the selfconsistent electric field $\vec{E}^* = \vec{E}^*(\vec{x}^*, t^*)$, and the selfconsistent magnetic field $\vec{B}^* = \vec{B}^*(\vec{x}^*, t^*)$ satisfy the following three-dimensional (3-D) Vlasov-Maxwell system (VMS):

$$
\begin{aligned}
&\frac{\partial f_1^\alpha}{\partial t} + \vec{w} \cdot \nabla_{\vec{x}} f_1^\alpha + \frac{q_\alpha}{m_\alpha} [\vec{E} + \vec{w} \times \vec{B}] \cdot \nabla_{\vec{w}} f_1^\alpha = 0 \\
&\qquad\qquad \text{for any } \alpha = 1, \ldots, \sigma, \\
&\nabla_{\vec{x}} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \sum_{\alpha=1}^{\sigma} q_\alpha \int_{\mathbb{R}^3} \vec{w} f_1^\alpha(\vec{x}, \vec{w}, t)\, d^3 w, \\
&\nabla_{\vec{x}} \cdot \vec{E} = \sum_{\alpha=1}^{\sigma} q_\alpha \int_{\mathbb{R}^3} f_1^\alpha(\vec{x}, \vec{w}, t)\, d^3 w, \\
&\nabla_{\vec{x}} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \\
&\nabla_{\vec{x}} \cdot \vec{B} = 0
\end{aligned} \tag{6}
$$

with the dimensionless variables

$$
\vec{x} := \frac{\vec{x}^*}{L}, \quad \vec{w} := \frac{\vec{w}^*}{c}, \quad t := \frac{c}{L} t^*,
$$

$$
f_1^\alpha := \frac{|q_1^*|^2}{m_1^*} \frac{L^2 c}{\varepsilon_0} f_1^{\alpha*} \quad (\alpha = 1, \ldots, \sigma),
$$

$$
\vec{E} := \frac{|q_1^*|}{m_1^*} \frac{L}{c^2} \vec{E}^*, \quad \vec{B} := \frac{|q_1^*|}{m_1^*} \frac{L}{c} \vec{B}^*,
$$

where $L$ is an arbitrary constant with the dimension of length, $c$ the speed of light, $\varepsilon_0$ the electric permittivity of free space, $q_\alpha := q_\alpha^*/|q_1^*|$, and $m_\alpha := m_\alpha^*/m_1^*$ ($\alpha = 1, \ldots, \sigma$). (The Cartesian coordinates of $\vec{w}$, $\vec{E}$, $\vec{B}$ (relative to the basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$) are denoted by $w^x, w^y, w^z, E^x, E^y, E^z, B^x, B^y$, and $B^z$, respectively.)

The Lie point symmetries of the one-dimensional (1-D) VMS are well studied. The full Lie point symmetry group admitted by the 1-D VMS without external fields or a background was calculated 1978 by Taranov [7]. In [7] Taranov also discussed the results of a work of Baranov [25], who calculated in 1976 a symmetry group admitted by the 1-D VMS for an electron plasma with a fixed and homogeneous ion background through a formal approach to the infinite system of PDEs for the moments of the electron distribution function. In 1985 Roberts [10] derived the full Lie point symmetry group admitted by the 1-D VMS in case of a one-species plasma with a background and a multi-species plasma with $q_\alpha/m_\alpha \neq q_\beta/m_\beta$ for $\alpha \neq \beta$ ($\alpha, \beta = 1, \ldots, \sigma$), where also a homogeneous background was assumed.

If one looks at a plasma consisting of $\sigma > 1$ species, where $q_\alpha/m_\alpha \neq q_\beta/m_\beta$ for any $\beta \neq \alpha$ ($\alpha, \beta = 1, \ldots, \sigma$) holds, the real Lie algebra $\mathcal{G}$ of the full Lie point symmetry group $G = \exp(\mathcal{G})$ admitted by (6) is of dimension $r = d(\mathcal{G}) = \dim(\mathcal{G}) = 8$ and may be spanned by the following infinitesimal generators written in terms of Cartesian coordinates:

$$
\begin{aligned}
\boldsymbol{v}_1 &= \frac{\partial}{\partial x}, \quad \boldsymbol{v}_2 = \frac{\partial}{\partial y}, \quad \boldsymbol{v}_3 = \frac{\partial}{\partial z}, \quad \boldsymbol{v}_4 = \frac{\partial}{\partial t}, \\
\boldsymbol{v}_5 &= y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + w^y \frac{\partial}{\partial w^x} - w^x \frac{\partial}{\partial w^y} + E^y \frac{\partial}{\partial E^x} \\
&\quad - E^x \frac{\partial}{\partial E^y} + B^y \frac{\partial}{\partial B^x} - B^x \frac{\partial}{\partial B^y}, \\
\boldsymbol{v}_6 &= z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} + w^z \frac{\partial}{\partial w^x} - w^x \frac{\partial}{\partial w^z} + E^z \frac{\partial}{\partial E^x} \\
&\quad - E^x \frac{\partial}{\partial E^z} + B^z \frac{\partial}{\partial B^x} - B^x \frac{\partial}{\partial B^z}, \\
\boldsymbol{v}_7 &= z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} + w^z \frac{\partial}{\partial w^y} - w^y \frac{\partial}{\partial w^z} + E^z \frac{\partial}{\partial E^y} \\
&\quad - E^y \frac{\partial}{\partial E^z} + B^z \frac{\partial}{\partial B^y} - B^y \frac{\partial}{\partial B^z}, \\
\boldsymbol{v}_8 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + t \frac{\partial}{\partial t} \\
&\quad - 2 \sum_{\alpha=1}^{\sigma} f_1^\alpha \frac{\partial}{\partial f_1^\alpha} - E^x \frac{\partial}{\partial E^x} - E^y \frac{\partial}{\partial E^y} - E^z \frac{\partial}{\partial E^z} \\
&\quad - B^x \frac{\partial}{\partial B^x} - B^y \frac{\partial}{\partial B^y} - B^z \frac{\partial}{\partial B^z}.
\end{aligned} \tag{7}
$$

The commutator table for this basis (7) of the eight-dimensional real Lie algebra $\mathcal{G}$ is shown in Table 1, from which the structure constants $C_{jk}^i$ of $\mathcal{G}$, which are defined by $[\boldsymbol{v}_j, \boldsymbol{v}_k] = \sum_{i=1}^{r} C_{jk}^i \boldsymbol{v}_i$ ($i, j, k = 1, \ldots, r$), can be

Table 1. The commutator table for the eight-dimensional Lie algebra $\mathscr{G}$.

| $[\boldsymbol{v}_i, \boldsymbol{v}_j]$ | $\boldsymbol{v}_1$ | $\boldsymbol{v}_2$ | $\boldsymbol{v}_3$ | $\boldsymbol{v}_4$ | $\boldsymbol{v}_5$ | $\boldsymbol{v}_6$ | $\boldsymbol{v}_7$ | $\boldsymbol{v}_8$ |
|---|---|---|---|---|---|---|---|---|
| $\boldsymbol{v}_1$ | 0 | 0 | 0 | 0 | $-\boldsymbol{v}_2$ | $-\boldsymbol{v}_3$ | 0 | $\boldsymbol{v}_1$ |
| $\boldsymbol{v}_2$ | 0 | 0 | 0 | 0 | $\boldsymbol{v}_1$ | 0 | $-\boldsymbol{v}_3$ | $\boldsymbol{v}_2$ |
| $\boldsymbol{v}_3$ | 0 | 0 | 0 | 0 | 0 | $\boldsymbol{v}_1$ | $\boldsymbol{v}_2$ | $\boldsymbol{v}_3$ |
| $\boldsymbol{v}_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\boldsymbol{v}_4$ |
| $\boldsymbol{v}_5$ | $\boldsymbol{v}_2$ | $-\boldsymbol{v}_1$ | 0 | 0 | 0 | $\boldsymbol{v}_7$ | $-\boldsymbol{v}_6$ | 0 |
| $\boldsymbol{v}_6$ | $\boldsymbol{v}_3$ | 0 | $-\boldsymbol{v}_1$ | 0 | $-\boldsymbol{v}_7$ | 0 | $\boldsymbol{v}_5$ | 0 |
| $\boldsymbol{v}_7$ | 0 | $\boldsymbol{v}_3$ | $-\boldsymbol{v}_2$ | 0 | $\boldsymbol{v}_6$ | $-\boldsymbol{v}_5$ | 0 | 0 |
| $\boldsymbol{v}_8$ | $-\boldsymbol{v}_1$ | $-\boldsymbol{v}_2$ | $-\boldsymbol{v}_3$ | $-\boldsymbol{v}_4$ | 0 | 0 | 0 | 0 |

read. Obviously, the center $\mathscr{Z}(\mathscr{G}) := \{\boldsymbol{u} \in \mathscr{G} \mid [\boldsymbol{u}, \boldsymbol{v}] = 0 \ \forall \boldsymbol{v} \in \mathscr{G}\}$ is the null algebra 0, and the derived algebra $\mathscr{G}^{(1)} := [\mathscr{G}, \mathscr{G}]$ is $\mathscr{G}^{(1)}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_7)$. Here and in what follows, $\mathscr{H}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_j)$ denotes the subalgebra $\mathscr{H}$ of $\mathscr{G}$ spanned by the basis vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_j$.

The corresponding subgroups of symmetries of the Vlasov-Maxwell equations are:

(a) Space translations (T(3)):

$$G_j\colon (\vec{x}, \vec{w}, t, f_1^1, \ldots, f_1^\sigma, \vec{E}, \vec{B}) \mapsto (\vec{x} + \varepsilon_j \vec{e}_j, \vec{w}, t, f_1^1, \ldots, f_1^\sigma, \vec{E}, \vec{B}) \quad (\varepsilon_j \in \mathbb{R};\ j = 1, 2, 3)\,.$$

(b) Time translation:

$$G_4\colon (\vec{x}, \vec{w}, t, f_1^1, \ldots, f_1^\sigma, \vec{E}, \vec{B}) \mapsto (\vec{x}, \vec{w}, t + \varepsilon_4, f_1^1, \ldots, f_1^\sigma, \vec{E}, \vec{B}) \quad (\varepsilon_4 \in \mathbb{R})\,.$$

(c) The group

$$\mathrm{SO}(3, \mathbb{R})\colon (\vec{x}, \vec{w}, t, f_1^1, \ldots, f_1^\sigma, \vec{E}, \vec{B}) \mapsto (R\vec{x}, R\vec{w}, t, f_1^1, \ldots, f_1^\sigma, R\vec{E}, R\vec{B})$$

of simultaneous rotations in the components of $\vec{x}$, $\vec{w}$, $\vec{E}$, $\vec{B}$ with the $3 \times 3$ orthogonal matrix

$$R := \underbrace{\begin{pmatrix} \cos\varepsilon_5 & \sin\varepsilon_5 & 0 \\ -\sin\varepsilon_5 & \cos\varepsilon_5 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{=:\, R_5} \underbrace{\begin{pmatrix} \cos\varepsilon_6 & 0 & \sin\varepsilon_6 \\ 0 & 1 & 0 \\ -\sin\varepsilon_6 & 0 & \cos\varepsilon_6 \end{pmatrix}}_{=:\, R_6} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon_7 & \sin\varepsilon_7 \\ 0 & -\sin\varepsilon_7 & \cos\varepsilon_7 \end{pmatrix}}_{=:\, R_7}$$

which depends on the three real parameters $\varepsilon_j$ ($j = 5, 6, 7$), and the one-parameter subgroups

$$G_j\colon (\vec{x}, \vec{w}, t, f_1^1, \ldots, f_1^\sigma, \vec{E}, \vec{B}) \mapsto (R_j\vec{x}, R_j\vec{w}, t, f_1^1, \ldots, f_1^\sigma, R_j\vec{E}, R_j\vec{B}) \quad (j = 5, 6, 7)\,.$$

(d) Scale transformation:

$$G_8\colon (\vec{x}, \vec{w}, t, f_1^1, \ldots, f_1^\sigma, \vec{E}, \vec{B}) \mapsto (\vec{x}\exp(\varepsilon_8), \vec{w}, t\exp(\varepsilon_8), f_1^1\exp(-2\varepsilon_8), \ldots, f_1^\sigma\exp(-2\varepsilon_8), \vec{E}\exp(-\varepsilon_8), \vec{B}\exp(-\varepsilon_8)) \quad (\varepsilon_8 \in \mathbb{R})\,.$$

The full symmetry group $G$ is generated by these one-parameter transformation groups. Roberts announced in [26] a manuscript on an investigation of the Lie point symmetries admitted by the 3-D VMS and described verbally the structure of the symmetry group as given above.

Three additional discrete symmetries of the 3-D VMS are given by

$$\begin{aligned} S_1\colon & (\vec{x}, \vec{w}, t, f_1^1, \ldots, f_1^\sigma, \vec{E}, \vec{B}) \mapsto (\vec{x}, -\vec{w}, -t, f_1^1, \ldots, f_1^\sigma, \vec{E}, -\vec{B}), \\ S_2\colon & (\vec{x}, \vec{w}, t, f_1^1, \ldots, f_1^\sigma, \vec{E}, \vec{B}) \mapsto (-\vec{x}, -\vec{w}, t, f_1^1, \ldots, f_1^\sigma, -\vec{E}, \vec{B}), \\ S_3\colon & (\vec{x}, \vec{w}, t, f_1^1, \ldots, f_1^\sigma, \vec{E}, \vec{B}) \mapsto (-\vec{x}, \vec{w}, -t, f_1^1, \ldots, f_1^\sigma, -\vec{E}, -\vec{B}), \end{aligned} \tag{8}$$

where $S_1 \circ S_2 = S_3$ holds. For $j = 1, 2, 3$ the mapping $S_j \circ S_j$ is the identity

$$S_0\colon (\vec{x}, \vec{w}, t, f_1^1, \ldots, f_1^\sigma, \vec{E}, \vec{B}) \mapsto (\vec{x}, \vec{w}, t, f_1^1, \ldots, f_1^\sigma, \vec{E}, \vec{B}).$$

For given $\boldsymbol{u} = \sum_{k=1}^{r} \alpha_k \boldsymbol{v}_k \in \mathscr{G}$ the inner derivation $\mathbf{ad}(\boldsymbol{u})\colon \mathscr{G} \to \mathscr{G}^{(1)}$ maps $\boldsymbol{w} = \sum_{j=1}^{r} \beta_j \boldsymbol{v}_j \in \mathscr{G}$ to $\mathbf{ad}(\boldsymbol{u})(\boldsymbol{w}) = [\boldsymbol{w}, \boldsymbol{u}] = \sum_{i,j=1}^{r} \left( \sum_{k=1}^{r} \alpha_k C_{jk}^i \right) \beta_j \boldsymbol{v}_i$ with

Table 2. Adjoint representation of the Lie group $G$.

| $g_i$ \ $v_j$: $\mathbf{Ad}(g_i)(v_j)$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ |
|---|---|---|---|---|
| $g_1 := \exp(\varepsilon_1 v_1)$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ |
| $g_2 := \exp(\varepsilon_2 v_2)$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ |
| $g_3 := \exp(\varepsilon_3 v_3)$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ |
| $g_4 := \exp(\varepsilon_4 v_4)$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ |
| $g_5 := \exp(\varepsilon_5 v_5)$ | $v_1 \cos\varepsilon_5 - v_2 \sin\varepsilon_5$ | $v_1 \sin\varepsilon_5 + v_2 \cos\varepsilon_5$ | $v_3$ | $v_4$ |
| $g_6 := \exp(\varepsilon_6 v_6)$ | $v_1 \cos\varepsilon_6 - v_3 \sin\varepsilon_6$ | $v_2$ | $v_1 \sin\varepsilon_6 + v_3 \cos\varepsilon_6$ | $v_4$ |
| $g_7 := \exp(\varepsilon_7 v_7)$ | $v_1$ | $v_2 \cos\varepsilon_7 - v_3 \sin\varepsilon_7$ | $v_2 \sin\varepsilon_7 + v_3 \cos\varepsilon_7$ | $v_4$ |
| $g_8 := \exp(\varepsilon_8 v_8)$ | $v_1 \exp\varepsilon_8$ | $v_2 \exp\varepsilon_8$ | $v_3 \exp\varepsilon_8$ | $v_4 \exp\varepsilon_8$ |

| $g_i$ \ $v_j$: $\mathbf{Ad}(g_i)(v_j)$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ |
|---|---|---|---|---|
| $g_1 := \exp(\varepsilon_1 v_1)$ | $v_5 + \varepsilon_1 v_2$ | $v_6 + \varepsilon_1 v_3$ | $v_7$ | $v_8 - \varepsilon_1 v_1$ |
| $g_2 := \exp(\varepsilon_2 v_2)$ | $v_5 - \varepsilon_2 v_1$ | $v_6$ | $v_7 + \varepsilon_2 v_3$ | $v_8 - \varepsilon_2 v_2$ |
| $g_3 := \exp(\varepsilon_3 v_3)$ | $v_5$ | $v_6 - \varepsilon_3 v_1$ | $v_7 - \varepsilon_3 v_2$ | $v_8 - \varepsilon_3 v_3$ |
| $g_4 := \exp(\varepsilon_4 v_4)$ | $v_5$ | $v_6$ | $v_7$ | $v_8 - \varepsilon_4 v_4$ |
| $g_5 := \exp(\varepsilon_5 v_5)$ | $v_5$ | $v_6 \cos\varepsilon_5 - v_7 \sin\varepsilon_5$ | $v_7 \cos\varepsilon_5 + v_6 \sin\varepsilon_5$ | $v_8$ |
| $g_6 := \exp(\varepsilon_6 v_6)$ | $v_7 \sin\varepsilon_6 + v_5 \cos\varepsilon_6$ | $v_6$ | $v_7 \cos\varepsilon_6 - v_5 \sin\varepsilon_6$ | $v_8$ |
| $g_7 := \exp(\varepsilon_7 v_7)$ | $v_5 \cos\varepsilon_7 - v_6 \sin\varepsilon_7$ | $v_5 \sin\varepsilon_7 + v_6 \cos\varepsilon_7$ | $v_7$ | $v_8$ |
| $g_8 := \exp(\varepsilon_8 v_8)$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ |

$$\mathbf{ad}_u := \left(\sum_{k=1}^{r=8} \alpha_k C^i_{jk}\right) = \begin{pmatrix} \alpha_8 & \alpha_5 & \alpha_6 & 0 & -\alpha_2 & -\alpha_3 & 0 & -\alpha_1 \\ -\alpha_5 & \alpha_8 & \alpha_7 & 0 & \alpha_1 & 0 & -\alpha_3 & -\alpha_2 \\ -\alpha_6 & -\alpha_7 & \alpha_8 & 0 & 0 & \alpha_1 & \alpha_2 & -\alpha_3 \\ 0 & 0 & 0 & \alpha_8 & 0 & 0 & 0 & -\alpha_4 \\ 0 & 0 & 0 & 0 & 0 & \alpha_7 & -\alpha_6 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_7 & 0 & \alpha_5 & 0 \\ 0 & 0 & 0 & 0 & \alpha_6 & -\alpha_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

The maximal rank of the endomorphism $\mathbf{ad}(u): \mathcal{G} \to \mathcal{G}^{(1)}$, obtained when $u$ ranges over the whole space $\mathcal{G}$, is $\max_{u \in \mathcal{G}}(\operatorname{rank}(\mathbf{ad}(u))) = 6$. As stated in (2), the general element in the group $\operatorname{Int}(\mathcal{G})$ may be given by

$$\mathbf{Ad}(g = \exp(\varepsilon_1 v_1) \cdot \ldots \cdot \exp(\varepsilon_8 v_8)) = \mathbf{Ad}(\exp(\varepsilon_1 v_1)) \circ \ldots \circ \mathbf{Ad}(\exp(\varepsilon_8 v_8))$$

with real parameters $\varepsilon_1, \ldots, \varepsilon_8$, where the inner automorphisms

$$\mathbf{Ad}(\exp(\varepsilon_j v_j)) := \exp(\varepsilon_j \, \mathbf{ad}(v_j)): \mathcal{G} \to \mathcal{G} \quad (j = 1, \ldots, 8)$$

can be computed as Lie series (cf. (3)) in the usual way and are listed in Table 2.

The computation of the Killing form $K_{\mathcal{G}}: (u, w) \mapsto K_{\mathcal{G}}(u, w) := \operatorname{tr}(\mathbf{ad}(u) \circ \mathbf{ad}(w))$ and the Casimir polynomial $C_{\mathcal{G}}: u \mapsto C_{\mathcal{G}}(u) := K_{\mathcal{G}}(u, u)$ gives $K_{\mathcal{G}}(u, w) = 4(-\alpha_5\beta_5 - \alpha_6\beta_6 - \alpha_7\beta_7 + \alpha_8\beta_8)$ and $C_{\mathcal{G}}(u) = 4(-\alpha_5^2 - \alpha_6^2 - \alpha_7^2 + \alpha_8^2)$. By applying Cartan's criteria of solvability (see e.g. Sect. 1 in I) and semisimplicity it follows that $\mathcal{G}$ is neither solvable nor semisimple. With the aid of theorem 2 one finds that the radical $\mathcal{G}$ is $\mathcal{R}(\mathcal{G})(v_1, v_2, v_3, v_4, v_8)$. A Levi factor of the Lie algebra $\mathcal{G}$ is the semisimple (even simple) rotation algebra $\mathcal{SO}(3, \mathbb{R})$ spanned by the basis $\{v_5, v_6, v_7\}$. Thus, a Levi decomposition of $\mathcal{G}$ is given by the semidirect sum $\mathcal{G} = \mathcal{R}(\mathcal{G}) \oplus_s \mathcal{SO}(3, \mathbb{R})$.

The total number $\ell_*(\mathcal{G})$ of all functionally independent scalar invariants of an arbitrary $u = \sum_{j=1}^{r} \alpha_j v_j \in \mathcal{G}$ (relative to the group $\operatorname{Int}(\mathcal{G})$) is

$$\ell_*(\mathcal{G}) = \mathrm{d}(\mathcal{G}) - \max_{u \in \mathcal{G}}(\operatorname{rank}(\mathbf{ad}(u))) = 8 - 6 = 2 \tag{10}$$

Table 3. (Bi-)linear invariants of a vector $\boldsymbol{u}$ in a subalgebra $\mathscr{H}$ of $\mathscr{G}$ (relative to $\mathrm{Int}(\mathscr{H})$).

| | Case | Subalgebra $\mathscr{H}$ | $d(\mathscr{H})$ | $\ell(\mathscr{H})$ | $\ell_*(\mathscr{H})$ | Invariants of $\boldsymbol{u}$ |
|---|---|---|---|---|---|---|
| 1 | | $\mathscr{H}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_8) = \mathscr{G}$ | 8 | 2 | 2 | $\alpha_5^2 + \alpha_6^2 + \alpha_7^2, \alpha_8$ |
| 2 | $\lambda_1(\boldsymbol{u}) = 0$ $(\alpha_8 = 0)$ | $\mathscr{H}_{\lambda_1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_7) = \mathscr{G}^{(1)}$ | 7 | 3 | 3 | $\alpha_4, \alpha_5^2 + \alpha_6^2 + \alpha_7^2,$ $\alpha_1, \alpha_7 - \alpha_2\alpha_6 + \alpha_3\alpha_5$ |
| 3 | $\mu_1(\boldsymbol{u}) = 0$ $(\alpha_5 = \alpha_6 = \alpha_7 = 0)$ | $\mathscr{H}_{\mu_1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_4, \boldsymbol{v}_8) = \mathscr{R}(\mathscr{G})$ | 5 | 1 | 1 | $\alpha_8$ |
| 4 | $\mu_2(\boldsymbol{u}) = 0$ $(\alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 0)$ | $\mathscr{H}_{\mu_2}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_4) = \mathscr{N}(\mathscr{G}) = \mathscr{H}_{\lambda_1} \cap \mathscr{H}_{\mu_1}$ | 4 | 4 | 4 | $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ |

(see (12) in I). The general invariant bilinear symmetric form

$$\Phi^{\mathscr{G}}: \mathscr{G} \times \mathscr{G} \to \mathbb{R}; (\boldsymbol{u}, \boldsymbol{w}) \mapsto \Phi^{\mathscr{G}}(\boldsymbol{u}, \boldsymbol{w}) := \sum_{j,k=1}^{r=8} \alpha_j \tilde{\Phi}_{jk}^{\mathscr{G}} \beta_k$$

with $\boldsymbol{w} = \sum_{k=1}^{r} \beta_k \boldsymbol{v}_k \in \mathscr{G}$ and the real symmetric matrix $(\tilde{\Phi}_{jk}^{\mathscr{G}}) = \mathrm{diag}(0, 0, 0, 0, b_2, b_2, b_2, b_1)$ can be determined from the infinitesimal criterion (11) in I (using the program OPTSYS), where $b_1$, $b_2$ are free real constants. (The Killing form is such an invariant bilinear form with $b_1 = -b_2 = 4$.) From the expression $\Phi^{\mathscr{G}}(\boldsymbol{u}, \boldsymbol{u})$ for an arbitrary $\boldsymbol{u} = \sum_{j=1}^{r} \alpha_j \boldsymbol{v}_j \in \mathscr{G}$ we find the linear invariant $\alpha_8$ (corresponding to the invariant bilinear form $\Phi^{(1)}$ with $\Phi^{(1)}(\boldsymbol{u}, \boldsymbol{u}) := \alpha_8^2$) and the "bilinear" invariant $a^2$ with $a := \sqrt{\alpha_5^2 + \alpha_6^2 + \alpha_7^2}$ (corresponding to the invariant bilinear form $\Phi^{(2)}$ with $\Phi^{(2)}(\boldsymbol{u}, \boldsymbol{u}) := a^2$) of the vector $\boldsymbol{u}$ (relative to the real group $\mathrm{Int}(\mathscr{G})$), which can be regarded as the $\ell_*(\mathscr{G}) = 2$ functionally independent scalar invariants of $\boldsymbol{u}$ (see also Table 3).

From the Killing polynomial $k_{\mathscr{G}}$, which is the characteristic polynomial of the inner derivation $\mathbf{ad}(\boldsymbol{u})$ for $\boldsymbol{u} \in \mathscr{G}$, the rank $\ell(\mathscr{G})$ of $\mathscr{G}$, which is defined as the minimal (algebraic) multiplicity of the eigenvalue 0 of $\mathbf{ad}(\boldsymbol{u})$ for all the vector fields $\boldsymbol{u} \in \mathscr{G}$, can be read (see e.g. Sect. 1 in I) and one finds $\ell(\mathscr{G}) = 2$. Furthermore, the Killing polynomial for $\boldsymbol{u} = \sum_{j=1}^{r} \beta_j \boldsymbol{v}_j \in \mathscr{G}$ has at most the $d(\mathscr{G}) - \ell(G) = 6$ nonzero roots $\lambda_1(\boldsymbol{u}) = \alpha_8 \in \mathbb{R}$ with multiplicity 2, $\mu_1(\boldsymbol{u}) = ia \in \mathbb{C}$, $\overline{\mu_1(\boldsymbol{u})} = -ia \in \mathbb{C}$, $\mu_2(\boldsymbol{u}) = \alpha_8 + ia \in \mathbb{C}$, $\overline{\mu_2(\boldsymbol{u})} = \alpha_8 - ia \in \mathbb{C}$. These eigenvalues of the linear mapping $\mathbf{ad}(\boldsymbol{u})$ are scalar invariants of $\boldsymbol{u} \in \mathscr{G}$ (see I). Obviously, $\lambda_1(\boldsymbol{u})$ is a linear invariant of $\boldsymbol{u}$ (relative to $\mathrm{Int}(\mathscr{G})$). Thus the set $\mathscr{H}_{\lambda_1} := \{\boldsymbol{w} \in \mathscr{G} \mid \lambda_1(\boldsymbol{w}) = 0\} = \mathscr{G}^{(1)}$ forms an ideal in $\mathscr{G}$ (see Sect. 1 in I). Since $a^2 = \alpha_5^2 + \alpha_6^2 + \alpha_7^2$ is a positive definite bilinear form in the coordinates $\alpha_5, \alpha_6, \alpha_7$ of $\boldsymbol{u}$, $\mathscr{H}_{\mu_1} := \{\boldsymbol{w} \in \mathscr{G} \mid \mu_1(\boldsymbol{u}) = 0\}$ is also an ideal of $\mathscr{G}$ (see Sect. 1 in I). $\mathscr{H}_{\mu_1}$ coincides with the radical $\mathscr{R}(\mathscr{G})$ of $\mathscr{G}$ $\mathscr{H}_{\mu_2} := \{\boldsymbol{w} \in \mathscr{G} \mid \mu_2(\boldsymbol{w}) = 0\}$ as the intersection of the ideals $\mathscr{H}_{\lambda_1}$ and $\mathscr{H}_{\mu_1}$ forms also an ideal. The abelian ideal $\mathscr{H}_{\mu_2}$ is the nilradical $\mathscr{N}(\mathscr{G})$ of $\mathscr{G}$. For these ideals $\mathscr{H}$ the (bi-)linear invariants of an arbitrary element $\boldsymbol{u} = \sum_{j=1}^{r} \alpha_j \boldsymbol{v}_j \in \mathscr{H}$ relative to the group $\mathrm{Int}(\mathscr{H})$ of the inner automorphisms $\mathscr{H} \to \mathscr{H}$ were calculated (using the REDUCE 3.2 program BINV). They are listed in Table 3, where the dimensionality, the rank and the total number of functionally independent scalar invariants (cf. (10)) for these ideals $\mathscr{H}$ are also shown.

According to the techniques for the construction of an optimal subalgebraic system $\Theta_1^{\mathscr{G}}$ for the Lie algebra $\mathscr{G}$ described in Sect. 1 in I, our aim during the classification process is to shorten the search for the conjugacy classes in $\Theta_1^{\mathscr{G}}$ by separating all the one-dimensional subalgebras of $\mathscr{G}$ in nonintersecting algebra classes with the aid of the calculated invariant bilinear forms (relative to the inner automorphisms group $\mathrm{Int}(\mathscr{G})$). Since these classes are invariant relative to all inner automorphisms $\mathscr{G} \to \mathscr{G}$, subalgebras of different classes cannot be conjugate (see subsection 1.1 in I). In order to determine these invariant algebra classes, we first choose a nonzero vector $\boldsymbol{u} = \sum_{j=1}^{r} \alpha_j \boldsymbol{v}_j \in \mathscr{G}$ with arbitrary coefficients $\alpha_1, \ldots, \alpha_r$, which spans the one-dimensional subalgebra $\mathscr{H}(\boldsymbol{u})$, and consider the (bi-)linear invariants of $\boldsymbol{u}$. Then we try to "simplify" the basis vector $\boldsymbol{u}$ of a representative subalgebra $\mathscr{H}(\boldsymbol{u})$ for any of the determined algebra classes by calculating a conjugate element

$$\boldsymbol{w} = \sum_{j=1}^{r} \tilde{\alpha}_j \boldsymbol{v}_j := \mathbf{Ad}(g = \exp(\varepsilon_1 \boldsymbol{v}_1) \cdot \ldots \cdot \exp(\varepsilon_r \boldsymbol{v}_r))(\boldsymbol{u})$$

with proper values for the real numbers $\varepsilon_1, \dots, \varepsilon_r$ such that as much coefficients $\tilde{\alpha}_1, \dots, \tilde{\alpha}_r$ vanish as possible. The various alternative cases give the conjugacy classes of conjugate subalgebras and, from them, an optimal system $\Theta_1^{\mathscr{G}}$ for $\mathscr{G}$ (see subsection 1.2 in I).

Since $\Phi^{(1)}$ with $\Phi^{(1)}(\boldsymbol{u}, \boldsymbol{u}) = \alpha_8^2$ is an invariant bilinear form, any one-dimensional subalgebra $\mathscr{H}(\boldsymbol{u})$ of $\mathscr{G}$ belongs to one of the two nonintersecting algebra classes given by $\Phi^{(1)}(\boldsymbol{u}, \boldsymbol{u}) > 0$ and $\Phi^{(1)}(\boldsymbol{u}, \boldsymbol{u}) = 0$, respectively. Thus, we only must investigate subalgebras $\mathscr{H}(\boldsymbol{u})$ of $\mathscr{G}$ with either $\boldsymbol{u} = \sum_{j=1}^{7} \alpha_j \boldsymbol{v}_j + \boldsymbol{v}_8$ (For the sake of simplicity we choose $\alpha_8 = 1$.) or $\boldsymbol{u} = \sum_{j=1}^{7} \alpha_j \boldsymbol{v}_j \neq 0$. Each of these two algebra classes can be further separated in the two classes with $\Phi^{(2)}(\boldsymbol{u}, \boldsymbol{u}) = a^2 := \alpha_5^2 + \alpha_6^2 + \alpha_7^2 > 0$ and $\Phi^{(2)}(\boldsymbol{u}, \boldsymbol{u}) = a^2 = 0$ ($\alpha_5 = \alpha_6 = \alpha_7 = 0$), respectively. Hence, we concentrate on the four nonintersecting algebra classes

$$A\colon\ \alpha_8 = 1 \quad \text{and} \quad a > 0,$$

$$B\colon\ \alpha_8 = 0 \quad \text{and} \quad a > 0,$$

$$C\colon\ \alpha_8 = 1 \quad \text{and} \quad a = 0,$$

$$D\colon\ \alpha_8 = 0 \quad \text{and} \quad a = 0.$$

(From the Casimir polynomial $C_{\mathscr{G}}$ with $C_{\mathscr{G}}(\boldsymbol{u}) = K_{\mathscr{G}}(\boldsymbol{u}, \boldsymbol{u}) = 4(-a^2 + \alpha_8^2)$ it follows only a separation of all the one-dimensional subalgebras $\mathscr{H}(\boldsymbol{u})$ into the three algebra classes given by $C_{\mathscr{G}}(\boldsymbol{u}) > 0$ ($\alpha_8^2 < a^2$), $C_{\mathscr{G}}(\boldsymbol{u}) = 0$ ($\alpha_8^2 = a^2$) and $C_{\mathscr{G}}(\boldsymbol{u}) > 0$ ($\alpha_8^2 > a^2$).) In addition, we investigate the vector $\hat{\boldsymbol{u}} = \sum_{j=1}^{8} \hat{\alpha}_j \boldsymbol{v}_j = (\mathbf{Ad}(\exp(\varepsilon_6 \boldsymbol{v}_6)) \circ \mathbf{Ad}(\exp(\varepsilon_7 \boldsymbol{v}_7)))(\boldsymbol{u})$ for $\boldsymbol{u} = \sum_{j=1}^{8} \alpha_j \boldsymbol{v}_j \neq 0$ with $\alpha_8 = 1$ or $\alpha_8 = 0$. If we choose

$$\varepsilon_7 = \begin{cases} \arctan\left(\dfrac{\alpha_6}{\alpha_5}\right) & \text{for} \quad \alpha_5 \neq 0 \\ \dfrac{\pi}{2}\operatorname{sign}(\alpha_6) & \text{for} \quad \alpha_5 = 0 \text{ and } \alpha_6 \neq 0 \\ \mp \arctan\left(\dfrac{\alpha_2}{\alpha_3}\right) & \text{for} \quad \alpha_3 \neq 0 \text{ and } \alpha_5 = \alpha_6 = \alpha_7 = 0 \\ \mp \dfrac{\pi}{2}\operatorname{sign}(\alpha_2) & \text{for} \quad \alpha_2 \neq 0 \text{ and } \alpha_3 = \alpha_5 = \alpha_6 = \alpha_7 = 0 \end{cases}$$

(The upper (lower) sign is choosen for $\alpha_4 \geq 0$ ($\alpha_4 < 0$).) and

$$\varepsilon_6 = \begin{cases} -\arctan\left(\dfrac{\alpha_7}{\sqrt{\alpha_5^2 + \alpha_6^2}}\right) & \text{for} \quad \sqrt{\alpha_5^2 + \alpha_6^2} > 0 \\ -\dfrac{\pi}{2}\operatorname{sign}(\alpha_7) & \text{for} \quad \alpha_5 = \alpha_6 = 0 \text{ and } \alpha_7 \neq 0 \\ -\arctan\left(\dfrac{\alpha_1}{\sqrt{\alpha_2^2 + \alpha_3^2}}\right) & \text{for} \quad \sqrt{\alpha_2^2 + \alpha_3^2} > 0 \text{ and } \alpha_5 = \alpha_6 = \alpha_7 = 0 \\ -\dfrac{\pi}{2}\operatorname{sign}(\alpha_1) & \text{for} \quad \alpha_2 = \alpha_3 = \alpha_5 = \alpha_6 = \alpha_7 = 0 \end{cases}$$

we find that $\boldsymbol{u} = \sum_{j=1}^{8} \alpha_j \boldsymbol{v}_j \neq 0$ with $\alpha_8 = 1$ or $\alpha_8 = 0$ is conjugate either to $\hat{\boldsymbol{u}} = \sum_{j=1}^{3} \hat{\alpha}_j \boldsymbol{v}_j + \alpha_4 \boldsymbol{v}_4 + a \boldsymbol{v}_5 + \alpha_8 \boldsymbol{v}_8$ for $a > 0$ or to $\hat{\boldsymbol{u}} = \pm\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}\, \boldsymbol{v}_3 + \alpha_4 \boldsymbol{v}_4 + \alpha_8 \boldsymbol{v}_8$ for $a = \sqrt{\alpha_5^2 + \alpha_6^2 + \alpha_7^2} = 0$, where the upper (lower) sign holds for $\alpha_4 \geq 0$ ($\alpha_4 < 0$). Thus, we can restrict our investigation to the four classes $A$, $B$, $C$, $D$ of one-dimensional subalgebras $\mathscr{H}(\boldsymbol{u})$ of $\mathscr{G}$ spanned by

$$\boldsymbol{u} = \begin{cases} \boldsymbol{u}_A := \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3 + \alpha_4 \boldsymbol{v}_4 + \alpha_5 \boldsymbol{v}_5 + \boldsymbol{v}_8 \quad \text{with} \quad \alpha_5 > 0 \\ \quad \text{or} \\ \boldsymbol{u}_B := \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3 + \alpha_4 \boldsymbol{v}_4 + \boldsymbol{v}_5 \\ \quad \text{or} \\ \boldsymbol{u}_C := \alpha_3 \boldsymbol{v}_3 + \alpha_4 \boldsymbol{v}_4 + \boldsymbol{v}_8 \quad \text{with} \quad \alpha_3 \geq 0 \ (\alpha_3 < 0) \quad \text{for} \quad \alpha_4 \geq 0 \ (\alpha_4 < 0) \\ \quad \text{or} \\ \boldsymbol{u}_D := \alpha_3 \boldsymbol{v}_3 + \alpha_4 \boldsymbol{v}_4 \neq 0 \quad \text{with} \quad \alpha_3 \geq 0 \ (\alpha_3 < 0) \quad \text{for} \quad \alpha_4 \geq 0 \ (\alpha_4 < 0) \end{cases}.$$

Since any one-dimensional subalgebra $\mathscr{K}(\boldsymbol{v})$ of $\mathscr{G}$ must be conjugate (relative to $\mathrm{Int}(\mathscr{G})$) to one of the four nonintersecting classes $A$, $B$, $C$, $D$, which are invariant relative to $\mathrm{Int}(\mathscr{G})$, we only must consider the four classes independently:

*A*) Any vector $\boldsymbol{u}_A$ is a regular element (see Sect. 1 of I) of $\mathscr{G}$, for which the number of distinct roots of the Killing polynomial is maximized. Furthermore, $\boldsymbol{u}_A$ is conjugate to the regular element $\boldsymbol{w}_{1a} := \tilde{\alpha}_5 \boldsymbol{v}_5 + \boldsymbol{v}_8$ of $\mathscr{G}$ with $\tilde{\alpha}_5 > 0$, since the inner automorphism

$$\mathbf{Ad}\left(\exp\left(\frac{\alpha_1 - \alpha_2 \alpha_5}{1 + \alpha_5^2} \boldsymbol{v}_1\right)\right) \circ \mathbf{Ad}\left(\exp\left(\frac{\alpha_2 + \alpha_1 \alpha_5}{1 + \alpha_5^2} \boldsymbol{v}_2\right)\right)$$
$$\circ\, \mathbf{Ad}(\exp(\alpha_3 \boldsymbol{v}_3)) \circ \mathbf{Ad}(\exp(\alpha_4 \boldsymbol{v}_4))$$

maps $\boldsymbol{u}_A$ to $\boldsymbol{w}_{1a}$ with $\alpha_5 = \tilde{\alpha}_5$.

*B*) As Case 2 in Table 3 indicates, the scalar invariants of $\boldsymbol{u}_B \in \mathscr{H}_{\lambda_1} = \mathscr{G}^{(1)}$ (relative to $\mathrm{Int}(\mathscr{H}_{\lambda_1})$) are $\alpha_4$, $\alpha_5 = 1$ and $\alpha_3 \alpha_5 = \alpha_3$. By applying the inner automorphism $\mathbf{Ad}(\exp(\varepsilon_1 \boldsymbol{v}_1)) \circ \mathbf{Ad}(\exp(\varepsilon_2 \boldsymbol{v}_2))$ with the real coefficients $\varepsilon_1 = (\alpha_1 - \alpha_2)/2$ and $\varepsilon_2 = (\alpha_2 + \alpha_1)/2$ to $\boldsymbol{u}_B$, one finds that $\boldsymbol{u}_B$ is conjugate to $\hat{\boldsymbol{u}}_B = \alpha_3 \boldsymbol{v}_3 + \alpha_4 \boldsymbol{v}_4 + \boldsymbol{v}_5$. If $\alpha_4 = 0$, $\hat{\boldsymbol{u}}_B = \alpha_3 \boldsymbol{v}_3 + \boldsymbol{v}_5$ is conjugate either to $\boldsymbol{w}_2 := \boldsymbol{v}_5$ for $\alpha_3 = 0$ or to

$$\mathbf{Ad}(\exp(-\log|\alpha_3|\, \boldsymbol{v}_8))(\alpha_3 \boldsymbol{v}_3 + \boldsymbol{v}_5)$$
$$= \pm(\boldsymbol{v}_3 \pm \boldsymbol{v}_5) =: \pm \boldsymbol{w}_3$$

for $\alpha_3 \neq 0$. In case of $\alpha_4 \neq 0$ one proves that the vector $\hat{\boldsymbol{u}}_B$ is conjugate to

$$\tilde{\boldsymbol{u}}_B := \mathbf{Ad}(\exp(-\log|\alpha_4|\, \boldsymbol{v}_8))(\tilde{\boldsymbol{u}}_B)$$
$$= \alpha_3/|\alpha_4|\, \boldsymbol{v}_3 + \operatorname{sign}(\alpha_4)\, \boldsymbol{v}_4 + \boldsymbol{v}_5 .$$

Since

$$\mathbf{Ad}(\exp(\pi \boldsymbol{v}_7))(\tilde{\alpha}_3 \boldsymbol{v}_3 - \boldsymbol{v}_4 + \boldsymbol{v}_5) = -(\tilde{\alpha}_3 \boldsymbol{v}_3 + \boldsymbol{v}_4 + \boldsymbol{v}_5),$$

it is possible to choose $\boldsymbol{w}_4 := \tilde{\alpha}_3 \boldsymbol{v}_3 + \boldsymbol{v}_4 + \boldsymbol{v}_5$ with $\tilde{\alpha}_3 \in \mathbb{R}$ as a representative for $\boldsymbol{u}_B$ in case of $\alpha_4 \neq 0$.

*C*) The only nonzero invariant of the regular element $\boldsymbol{u}_C$ in the ideal $\mathscr{R}(\mathscr{G}) = \mathscr{H}_{\mu_1}$ is its coefficient $\alpha_8 = 1$ (see case 3 in Table 3). The vector $\boldsymbol{u}_C$ is conjugate to $\boldsymbol{w}_{1b} := \boldsymbol{v}_8$, as is easily verified by using the inner automorphism $\mathbf{Ad}(\exp(\alpha_3 \boldsymbol{v}_3)) \circ \mathbf{Ad}(\exp(\alpha_4 \boldsymbol{v}_4))$.

*D*) The scalar invariants of $\boldsymbol{u}_D = \alpha_3 \boldsymbol{v}_3 + \alpha_4 \boldsymbol{v}_4 \in \mathscr{H}_{\mu_2} = \mathscr{N}(\mathscr{G})$ are $\alpha_3$ and $\alpha_4$, which can be read from case 4 in Table 3. In case $\alpha_4 \neq 0$, $\boldsymbol{u}_D$ is conjugate to $\boldsymbol{w}_5 := \tilde{\alpha}_3 \boldsymbol{v}_3 + \boldsymbol{v}_4$ with $\tilde{\alpha}_3 \geq 0$ (see the calculations for class $B$). If $\alpha_4 = 0$, the vector $\boldsymbol{u}_D$ is a multiple of $\boldsymbol{w}_6 := \boldsymbol{v}_3$.

So an optimal system $\Theta_1^{\mathscr{G}}$ for the Lie algebra $\mathscr{G}$ is the union of the one-dimensional subalgebras $\mathscr{H}(\boldsymbol{w}_1), \ldots, \mathscr{H}(\boldsymbol{w}_6)$, which are summarized in Table 4. (For simplicity the tilde of the coefficients $\tilde{\alpha}_j$ is dropped.) In this table the real coefficients $\alpha_3$ and $\alpha_5$ are nonnegative numbers, unless otherwise stated. (If $0 \leq \alpha_3 \neq \tilde{\alpha}_3 \geq 0$, the two subalgebras $\mathscr{H}(\alpha_3 \boldsymbol{v}_3 + \boldsymbol{v}_4)$ and $\mathscr{H}(\tilde{\alpha}_3 \boldsymbol{v}_3 + \boldsymbol{v}_4)$ stand for nonconjugate one-dimensional subalgebras of $\mathscr{G}$.) With regard to the discrete symmetries (8) of the 3-D VMS a further "simplification" of the members in $\Theta_1^{\mathscr{G}}$ is possible. For example, the subalgebras $\mathscr{H}(\boldsymbol{v}_3 \pm \boldsymbol{v}_5)$ may be replaced by $\mathscr{H}(\boldsymbol{v}_3 + \boldsymbol{v}_5)$, since $S_2(\boldsymbol{v}_3 - \boldsymbol{v}_5) = -(\boldsymbol{v}_3 + \boldsymbol{v}_5)$.

In order to construct an optimal subalgebraic system $\Theta_2^{\mathscr{G}}$ for $\mathscr{G}$ with the aid of the "method of expansion" of the members in $\Theta_1^{\mathscr{G}}$, one has to determine the normalizers $\mathbf{Nor}(\mathscr{H}(\boldsymbol{w}_j)) := \{\boldsymbol{v} \in \mathscr{G} \mid [\boldsymbol{v}, \boldsymbol{w}_j] \in \mathscr{H}(\boldsymbol{w}_j)\}$ of

Table 4. Optimal system $\Theta_1^{\mathscr{G}}$ of one-dimensional subalgebras ($\alpha_3, \alpha_5 \geq 0$).

| | | |
|---|---|---|
| $\mathscr{H}(\boldsymbol{v}_3)$ | $\mathscr{H}(\alpha_3 \boldsymbol{v}_3 + \boldsymbol{v}_4)$ | $\mathscr{H}(\alpha_3 \boldsymbol{v}_3 + \boldsymbol{v}_4 + \boldsymbol{v}_5)$ with $\alpha_3 \in \mathbb{R}$ |
| $\mathscr{H}(\boldsymbol{v}_3 \pm \boldsymbol{v}_5)$ | $\mathscr{H}(\boldsymbol{v}_5)$ | $\mathscr{H}(\alpha_5 \boldsymbol{v}_5 + \boldsymbol{v}_8)$ |

Table 5. Normalizers of the subalgebras $\mathscr{H}$ in $\Theta_1^{\mathscr{G}}$ and possible representatives $\mathscr{K}$ in $\Theta_2^{\mathscr{G}}$.

| $\mathscr{H}(\boldsymbol{u}) \in \Theta_1^{\mathscr{G}}$ | $\mathbf{Nor}(\mathscr{H}(\boldsymbol{u}))$ | $\mathscr{K}(\boldsymbol{u}, \boldsymbol{v})$ $\left(\boldsymbol{v} = \sum_{k=1}^{8} \beta_k \boldsymbol{v}_k \in \mathbf{Nor}(\mathscr{H}(\boldsymbol{u}))/\mathscr{H}(\boldsymbol{u})\right)$ |
|---|---|---|
| $\mathscr{H}(\boldsymbol{v}_3)$ | $\mathscr{H}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_5, \boldsymbol{v}_8)$ | $\mathscr{K}(\boldsymbol{v}_3, \beta_1 \boldsymbol{v}_1 + \beta_2 \boldsymbol{v}_2 + \beta_4 \boldsymbol{v}_4 + \beta_5 \boldsymbol{v}_5 + \beta_8 \boldsymbol{v}_8)$ |
| $\mathscr{H}(\boldsymbol{v}_4)$ | $\mathscr{G}$ | $\mathscr{K}\left(\boldsymbol{v}_4, \beta_1 \boldsymbol{v}_1 + \beta_2 \boldsymbol{v}_2 + \beta_3 \boldsymbol{v}_3 + \sum_{j=5}^{8} \beta_j \boldsymbol{v}_j\right)$ |
| $\mathscr{H}(\alpha_3 \boldsymbol{v}_3 + \boldsymbol{v}_4)$ $(\alpha_3 > 0)$ | $\mathscr{H}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_5, \boldsymbol{v}_8)$ | $\mathscr{K}(\alpha_3 \boldsymbol{v}_3 + \boldsymbol{v}_4, \beta_1 \boldsymbol{v}_1 + \beta_2 \boldsymbol{v}_2 + \beta_3 \boldsymbol{v}_3 + \beta_5 \boldsymbol{v}_5 + \beta_8 \boldsymbol{v}_8)$ |
| $\mathscr{H}(\alpha_3 \boldsymbol{v}_3 + \boldsymbol{v}_4 + \boldsymbol{v}_5)$ | $\mathscr{H}(\boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5)$ | $\mathscr{K}(\alpha_3 \boldsymbol{v}_3 + \boldsymbol{v}_4 + \boldsymbol{v}_5, \beta_3 \boldsymbol{v}_3 + \beta_5 \boldsymbol{v}_5)$ |
| $\mathscr{H}(\boldsymbol{v}_3 \pm \boldsymbol{v}_5)$ | $\mathscr{H}(\boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5)$ | $\mathscr{K}(\boldsymbol{v}_3 \pm \boldsymbol{v}_5, \beta_3 \boldsymbol{v}_3 + \beta_4 \boldsymbol{v}_4)$ |
| $\mathscr{H}(\boldsymbol{v}_5)$ | $\mathscr{H}(\boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5)$ | $\mathscr{K}(\boldsymbol{v}_5, \beta_3 \boldsymbol{v}_3 + \beta_4 \boldsymbol{v}_4)$ |
| $\mathscr{H}(\boldsymbol{v}_8)$ | $\mathscr{H}(\boldsymbol{v}_5, \ldots, \boldsymbol{v}_8)$ | $\mathscr{K}(\boldsymbol{v}_8, \beta_5 \boldsymbol{v}_5 + \beta_6 \boldsymbol{v}_6 + \beta_7 \boldsymbol{v}_7)$ |
| $\mathscr{H}(\alpha_5 \boldsymbol{v}_5 + \boldsymbol{v}_8)$ $(\alpha_5 > 0)$ | $\mathscr{H}(\boldsymbol{v}_5, \boldsymbol{v}_8)$ | $\mathscr{K}(\alpha_5 \boldsymbol{v}_5 + \boldsymbol{v}_8, \boldsymbol{v}_8)$ |

Table 6. Optimal system $\Theta_2^{\mathscr{G}}$ of two-dimensional subalgebras $(\alpha_1, \alpha_3, \alpha_5 \geq 0)$.

| | | |
|---|---|---|
| $\mathscr{H}(\boldsymbol{v}_1, \boldsymbol{v}_3)$ | $\mathscr{H}(\boldsymbol{v}_3, \alpha_1 \boldsymbol{v}_1 + \boldsymbol{v}_4)$ | $\mathscr{H}(\boldsymbol{v}_3, \boldsymbol{v}_4 + \boldsymbol{v}_5)$ |
| $\mathscr{H}(\boldsymbol{v}_3, \boldsymbol{v}_5)$ | $\mathscr{H}(\boldsymbol{v}_3, \alpha_5 \boldsymbol{v}_5 + \boldsymbol{v}_8)$ | $\mathscr{H}(\alpha_3 \boldsymbol{v}_3 + \boldsymbol{v}_4, \boldsymbol{v}_3 \pm \boldsymbol{v}_5)$ |
| $\mathscr{H}(\alpha_3 \boldsymbol{v}_3 + \boldsymbol{v}_4, \boldsymbol{v}_5)$ | $\mathscr{H}(\alpha_3 \boldsymbol{v}_3 + \boldsymbol{v}_4, \alpha_5 \boldsymbol{v}_5 + \boldsymbol{v}_8)$ | with $\alpha_3 \in \mathbb{R}$ |
| | $\mathscr{H}(\boldsymbol{v}_5, \boldsymbol{v}_8)$ | |

the subalgebras $\mathscr{H}(\boldsymbol{w}_j)$ in $\Theta_1^{\mathscr{G}}$ for $j = 1, \ldots, 6$. These normalizers are listed in Table 5 and were calculated using the REDUCE 3.2 program NORM, which was briefly described in Sect. 1 in I. In the next step during the classification process (see subsection 1.2 in I), we choose for every one-dimensional subalgebra $\mathscr{H}(\boldsymbol{u})$ in $\Theta_1^{\mathscr{G}}$ an arbitrary nonnull vector $\boldsymbol{v} = \sum_{k=1}^{8} \beta_k \boldsymbol{v}_k \in \mathbf{Nor}(\mathscr{H}(\boldsymbol{u}))/\mathscr{H}(\boldsymbol{u})$ $(\beta_1, \ldots, \beta_8 \in \mathbb{R})$ and list the resulting two-dimensional subalgebras $\mathscr{K}(\boldsymbol{u}, \boldsymbol{v})$ of $\mathscr{G}$ (see also Table 5). From the resulting list of solvable subalgebras $\mathscr{K}$ we delete the conjugate ones except of one representative for every conjugacy class. The remaining representatives form an optimal subalgebraic system $\Theta_2^{\mathscr{G}}$ for $\mathscr{G}$, which is given in Table 6.

Analogously to the above given construction of $\Theta_2^{\mathscr{G}} = \tilde{\Theta}_2^{\mathscr{G}}$, optimal subsystems $\tilde{\Theta}_s^{\mathscr{G}}$ of $s$-dimensional solvable subalgebras in $\mathscr{G}$ may be constructed for $3 \leq s \leq 7$ (cf. step 1 in subsection 1.1). Since there are no seven-dimensional solvable subalgebras of $\mathscr{G}$, $\tilde{\Theta}_7^{\mathscr{G}}$ is an empty list. As noticed above, $\mathscr{G}$ is the semidirect sum of its five-dimensional radical $\mathscr{R}(\mathscr{G})$ with basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_8\}$ and the semisimple three-dimensional Levi subalgebra $\mathscr{L} := \mathscr{SO}(3, \mathbb{R})$. Thus, an optimal subsystem $\hat{\Theta}_3^{\mathscr{L}}$ is given by the Levi factor $\mathscr{SO}(3, \mathbb{R})$ spanned by $\boldsymbol{v}_5, \boldsymbol{v}_6, \boldsymbol{v}_7$. Hence, all optimal subsystems $\hat{\Theta}_s^{\mathscr{L}}$ of $s$-dimensional semisimple subalgebras in $\mathscr{SO}(3, \mathbb{R})$ for $4 \leq s \leq 7$ are empty sets (cf. step 2 in subsection 1.1). According to step 3 in the subsection 1.1, there remains the task of classifying the $s$-dimensional subalgebras of $\mathscr{G}$ with non-trivial Levi decompositions by means of optimal subsystems $\hat{\Theta}_s^{\mathscr{G}}$ for $3 \leq s \leq 7$. Since the choosen Levi factor $\mathscr{L}$ has the dimensionality $\mathrm{d}(\mathscr{L}) = 3$, $\hat{\Theta}_3^{\mathscr{G}}$ is empty and optimal subsystems $\hat{\Theta}_s^{\mathscr{G}}$ for $4 \leq s \leq 7$ can be constructed by considering the conjugacy classes of $s$-dimensional subalgebras of type $\mathscr{H} = \mathscr{R}' \oplus_s \mathscr{SO}(3, \mathbb{R})$, where $\mathscr{R}'$ is a $(s-3)$-dimensional solvable subalgebra of $\mathscr{G}$ such that $\mathscr{R}' \cap \mathscr{SO}(3, \mathbb{R}) = 0$ and $[\mathscr{R}', \mathscr{SO}(3, \mathbb{R})] \subset \mathscr{R}'$ hold. Full optimal subalgebraic systems $\Theta_s^{\mathscr{G}}$ for the Lie algebra $\mathscr{G}$ of infinitesimal symmetries admitted by the 3-D VMS are shown in Table 7 for $s = 3, \ldots, 7$, where the real numbers $\alpha_3, \alpha_4, \alpha_5, \alpha_8$ are non-negative, unless otherwise stated. Clearly, $\Theta_8^{\mathscr{G}}$ contains only $\mathscr{G}$ itself.

From the full optimal subalgebraic systems $\Theta_s^{\mathscr{G}}$ for $\mathscr{G}$ the corresponding optimal systems $\Theta_s^{G}$ of $s$-parameter subgroups for the symmetry group $G = \exp(\mathscr{G})$ admitted by the 3-D VMS follow by exponentiating the infinitesimal generators, which span the choosen $s$-dimensional subalgebraic representative in $\Theta_s^{\mathscr{G}}$ $(1 \leq s \leq 8)$. In general, the calculation of reduced systems of IPDEs, which are associated to the subgroups in the optimal systems for $G$ containing the rotation group $\mathscr{SO}(3, \mathbb{R})$, fails (cf. Olver [4]). Therefore, one is only interested in the optimal subsystems $\tilde{\Theta}_s^{G}$ of solvable subgroups of $G$ for $s = 1, \ldots, 6$ during the classification of the similarity solutions of the 3-D VMS. Fur-

Table 7. Full optimal systems $\Theta_s^{\mathscr{G}}$ of $s$-dimensional subalgebras in $\mathscr{G}$ for $s = 3, \ldots, 7$.

| | | | |
|---|---|---|---|
| $\Theta_3^{\mathscr{G}}$ | $\tilde{\Theta}_3^{\mathscr{G}}$: | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \alpha_3 \mathbf{v}_3 + \mathbf{v}_4)$ |
| | | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \alpha_3 \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5)$ | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \pm \mathbf{v}_5)$ |
| | | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5 + \alpha_8 \mathbf{v}_8)$ | $\mathscr{H}(\mathbf{v}_1 + \alpha_4 \mathbf{v}_4, \mathbf{v}_2, \mathbf{v}_8)$ |
| | | $\mathscr{H}(\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5)$ | $\mathscr{H}(\mathbf{v}_3, \mathbf{v}_4, \alpha_5 \mathbf{v}_5 + \mathbf{v}_8)$ |
| | | $\mathscr{H}(\mathbf{v}_3 + \alpha_4 \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_8)$ | $\mathscr{H}(\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_8)$ |
| | $\hat{\Theta}_3^{\mathscr{S}}$: | $\mathscr{H}(\mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7) = \mathscr{SO}(3, \mathbb{R})$ | |
| $\Theta_4^{\mathscr{G}}$ | $\tilde{\Theta}_4^{\mathscr{G}}$: | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \mathscr{N}(\mathscr{G})$ | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 + \mathbf{v}_5)$ |
| | | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5)$ | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \alpha_5 \mathbf{v}_5 + \mathbf{v}_8)$ |
| | | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \alpha_3 \mathbf{v}_3 + \mathbf{v}_4, \mathbf{v}_3 \pm \mathbf{v}_5)$ | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \alpha_3 \mathbf{v}_3 + \mathbf{v}_4, \mathbf{v}_5)$ |
| | | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \alpha_3 \mathbf{v}_3 + \mathbf{v}_4, \alpha_5 \mathbf{v}_5 + \mathbf{v}_8)$ with $\alpha_3 \in \mathbb{R}$ | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5, \mathbf{v}_8)$ |
| | | $\mathscr{H}(\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_8)$ | |
| | $\hat{\Theta}_4^{\mathscr{G}}$: | $\mathscr{H}(\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7) = \mathscr{H}(\mathbf{v}_4) \oplus \mathscr{SO}(3, \mathbb{R})$ | |
| | | $\mathscr{H}(\mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7, \mathbf{v}_8) = \mathscr{H}(\mathbf{v}_8) \oplus \mathscr{SO}(3, \mathbb{R})$ | |
| $\Theta_5^{\mathscr{G}}$ | $\tilde{\Theta}_5^{\mathscr{G}}$: | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5)$ | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \alpha_5 \mathbf{v}_5 + \mathbf{v}_8)$ |
| | | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 + \mathbf{v}_5, \mathbf{v}_8)$ | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5, \mathbf{v}_8)$ |
| | | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \alpha_3 \mathbf{v}_3 + \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_8)$ | |
| | $\hat{\Theta}_5^{\mathscr{G}}$: | $\mathscr{H}(\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7, \mathbf{v}_8) = \mathscr{H}(\mathbf{v}_4, \mathbf{v}_8) \oplus \mathscr{SO}(3, \mathbb{R})$ | |
| $\Theta_6^{\mathscr{G}}$ | $\tilde{\Theta}_6^{\mathscr{G}}$: | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_8)$ | |
| | $\hat{\Theta}_6^{\mathscr{G}}$: | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7) = \mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \oplus_s \mathscr{SO}(3, \mathbb{R})$ | |
| $\Theta_7^{\mathscr{G}}$ | $\hat{\Theta}_7^{\mathscr{G}}$: | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7) = \mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \oplus_s \mathscr{SO}(3, \mathbb{R})$ | |
| | | $\mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7, \mathbf{v}_8) = \mathscr{H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_8) \oplus_s \mathscr{SO}(3, \mathbb{R})$ | |

thermore, no $s$-parameter subgroup in $\tilde{\Theta}_s^G$ leads to a reduced system in $n-s = 7-s$ independent variables if $s=5$ or $s=6$. So the task to classify the similarity solutions of the 3-D VMS can be carried out on the basis of the reduced systems of IPDEs, which are associated with the members of the optimal subalgebraic subsystems $\tilde{\Theta}_s^{\mathscr{G}}$ with $1 \leq s \leq 4$. Lack of space precludes pursuing this interesting investigation here, and the reader is refered to my PhD thesis (to appear in 1993).

## 3. Concluding Remarks

An effective, systematic means to classify the similarity solutions of a given system of PDEs (IPDEs), which admits a finite-dimensional Lie point symmetry group $G$ with its real Lie algebra $\mathscr{G}$, are optimal systems of the group-invariant solutions, since every other such solution can be derived from these systems. The problem of classifying the similarity solutions by means of optimal systems leads to that of finding the equivalence classes of subalgebras for the Lie algebra $\mathscr{G}$ under conjugation. Full optimal subalgebraic systems for $\mathscr{G}$ follow from these conjugacy classes. In my previous work I [11] and in this paper, the techniques for the classification process for the subalgebras of $\mathscr{G}$, which are well described by Ovsiannikov [1], Ibragimov [2], and Olver [4], were summarized, and further developments of these techniques based on the computer-aided calculation of the bilinear invariant forms for the real valued inner automorphisms $\mathscr{G} \to \mathscr{G}$ were presented. In comparison with the usual techniques to obtain optimal subalgebraic systems the refined technique using the invariance properties of these forms saves a lot of time during the classification process. The advantage of this modified method was demonstrated here by applying it to the eight-dimensional Lie algebra of infinitesimal symmetries admitted by the non-relativistic 3-D Vlasov-Maxwell equations for a multi-species plasma without a background and external fields.

A more detailed description of the techniques to obtain optimal systems of subalgebras for the finite-dimensional real Lie algebra of infinitesimal symmetries admitted by a given system of PDEs or IPDEs will be stated in my PhD thesis (in German). In addition, the modified method to construct optimal subalgebraic systems, which is based on the knowledge of complete sets of functionally independent scalar invariants relative to the real Lie group of the inner automorphisms, will be applied to the Lie algebra $\mathscr{G}$ of the Lie point symmetry group $\mathscr{G}$ admitted by the relativistic 3-D Vlasov-Maxwell equations for a multi-species plasma *.

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* *Note added in proof:* In subsection 1.2 in I the following statement is given: If $\boldsymbol{u}$ is a regular element of the finite-dimensional real Lie algebra $\mathscr{G}$, then there exists a conjugate regular element $\boldsymbol{w}$ of $\mathscr{G}$ such that all the coefficients of $\boldsymbol{w}$ which are functionally independent of the invariants of $\boldsymbol{w}$ (relative to $\mathrm{Int}(\mathscr{G})$) vanish. This is only correct under some additional conditions, which are fulfilled for all real Lie algebras under consideration.